Ordinary Differential Equations and Linear Systems
MECH 412 - *System Dynamics and Control*

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- To control a system (i.e., a plant), we need a model of it.
- Wait ... what's "a system"?

Definition

A **system** is a relationship between inputs and outputs.

$$u(t) \longrightarrow \boxed{\text{Plant (DE)}} \longrightarrow y(t) \qquad u(s) \longrightarrow \boxed{\quad G(s) \quad} \longrightarrow y(s)$$

- The input $u(t) \in \mathbb{R}$ (or $\mathbf{u}(t) \in \mathbb{R}^{n_u}$) is applied by an actuator.
- The output $y(t) \in \mathbb{R}$ (or $\mathbf{y}(t) \in \mathbb{R}^{n_y}$) is measured by a sensor.
- Depending on context, the system being discussed could be
  - the real physical system, or
  - a mathematical model of the physical system.
- To design a controller, a mathematical model of the real physical system to be controlled is needed.

- What do we mean when we say "a mathematical model"? A CAD model? A FEM model? A CFD model?
- We mean a mathematical model that **appropriately** describes the **physics** of the system and the relationship between the system **inputs** (actuators) and **outputs** (measurements).
  - "All models are wrong, but some are useful." — George Box (an economist).
- The input and the output are *causally* related through a differential equation (DE).
  - A causal system is a system where the output at the current time t depends on past and current inputs (but not on future inputs).
  - Examples:
    Causal: $\dot{q}(t) = q(t) + u(t), \;\; y(t) = q(t)$ and $\dot{q}(t) = q(t) + u(t - T), \;\; y(t) = q(t)$.
    Not causal: $\dot{q}(t) = q(t) + u(t + T), \;\; y(t) = q(t)$ for some $0 < T < \infty$.
  - All real-world systems (e.g., aircraft, RLC circuits, robots) are causal.
- In this course we will primarily be concerned with systems that are described by **linear, constant coefficient, ordinary differential equations (i.e., linear time invariant [LTI] systems)**.

$$u(t) \longrightarrow \boxed{\text{Plant (DE)}} \longrightarrow y(t) \qquad u(s) \longrightarrow \boxed{\quad G(s) \quad} \longrightarrow y(s)$$

Kinds of DEs

- In this course we will primarily work with differential equations (DEs) that are
  - **linear** (not nonlinear),
  - **constant coefficient** (not time-varying coefficient),
  - **ordinary** (not partial), and
  - **nonhomogeneous** (not homogeneous).
- For example, consider

$$\ddot{q}(t) + d\dot{q}(t) + kq(t) = b\dot{u}(t) + u(t), \qquad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \tag{1}$$
$$y(t) = c\dot{q}(t). \tag{2}$$

  The DE in (1) is a nonhomogeneous ordinary DE (ODE) with constant coefficients (i.e., $d, k, b, c \in \mathbb{R}$ are all constant). The output (measurement) is proportional to $\dot{q}(t)$.
- If $\dot{u}(t) = u(t) = 0$ for all time then (1) is a homogeneous ODE.
- The system is the relationship between the input $u(t)$ and the output $y(t)$, and is described by both (1) and (2).

- The DE

$$\ddot{q}(t) + d(t)\dot{q}(t) + k(t)q(t) = b(t)\dot{u} + u(t), \qquad q(0) = q_0, \;\; \dot{q}(0) = \dot{q}_0,$$

  is an ODE with time-varying coefficients.
- The DE

$$\ddot{q}(t) + d\dot{q}(t) + k_1 q(t) + k_2 q^3(t) = b\dot{u}(t) + u(t), \qquad q(0) = q_0, \;\; \dot{q}(0) = \dot{q}_0,$$

  is a nonlinear ODE.
  - From time to time, nonlinear ODEs will be dealt with …because everything is nonlinear to some degree.
- As another example,

$$\rho \frac{\partial^2 q(x,t)}{\partial t^2} + EI \frac{\partial^4 q(x,t)}{\partial x^4} = u(t)$$

  with appropriate boundary conditions (e.g., Dirichlet, Neumann, Robin) and ICs is a partial DE (PDE).
  - However, whenever we're confronted with a PDE, we just …

State-Space Form

- State-space form is just a more compact way of writing a system. It is also useful when numerically simulating ODEs.
- Consider

$$\ddot{q}(t) + d\dot{q}(t) + kq(t) = u(t), \qquad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0,$$
$$y(t) = c\dot{q}(t).$$

- We want to write this system in the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), u(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0, \\ y(t) &= g(\mathbf{x}(t), u(t)), \end{aligned}$$

  where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

- All we are doing is writing the *second-order scalar* ODE in a *first-order state-space form*.
- In general,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)). \end{aligned}$$

- For a second-order scalar ODE, the state-space form will have two states. Let

$$x_1(t) = q(t), \qquad x_2(t) = \dot{x}_1(t) = \dot{q}(t).$$

  - In general, a n^{th} order scalar ODE will have a state-space realization with n states.
- In terms of the states $x_1(t)$ and $x_2(t)$, the system can be written as

$$\begin{gathered} \dot{x}_2(t) + dx_2(t) + kx_1(t) = u(t), \qquad x_1(0) = x_{1,0}, \;\; x_2(0) = x_{2,0}, \\ y(t) = cx_2(t). \end{gathered}$$

- Isolating $\dot{x}_2(t)$ results in

$$\begin{aligned} \dot{x}_2(t) &= -dx_2(t) - kx_1(t) + u(t), \qquad x_1(0) = x_{1,0}, \;\; x_2(0) = x_{2,0}, \\ y(t) &= cx_2(t). \end{aligned}$$

- Together with $\dot{x}_1(t) = x_2(t)$ we have

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} x_2(t) \\ -dx_2(t) - kx_1(t) + u(t) \end{bmatrix} = \mathbf{f}(\mathbf{x}(t), u(t)), \qquad x_1(0) = x_{1,0}, \;\; x_2(0) = x_{2,0}, \\ y(t) &= cx_2(t) + 0u(t) = g(\mathbf{x}(t), u(t)). \end{aligned}$$

Linear - What does that mean, anyway?

How do we know or how can we determine if a DE or, more generally, a system is linear?

To answer the above question we first must understand what a linear function is.

A linear function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ must satisfy

- **superposition**,
$$\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2), \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n,$$
- and **scaling**,
$$\mathbf{f}(a\mathbf{x}) = a\mathbf{f}(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{R}^n, \forall a \in \mathbb{R}.$$

Together,

$$\mathbf{f}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1\mathbf{f}(\mathbf{x}_1) + a_2\mathbf{f}(\mathbf{x}_2), \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n,\ \forall a_1, a_2 \in \mathbb{R}.$$

Note that $f_{\text{lin.}}(x) = mx$ is linear in x, while $f_{\text{not lin.}}(x) = mx + b$ is not linear in x!

A Linear System

- Any system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0,\\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)).\end{aligned}$$

  where $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{g} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ both satisfy superposition and scaling is a **linear system**.

- Linear systems can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).\end{aligned}$$

- This is actually a linear time-invariant system (LTI) because $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are all constant matrices (that is, they are invariant to a change in time).
- The state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is **not** unique.

- To be clear, a linear system with

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),\end{aligned}$$

is "linear" because (where we drop "(t)" for convenience)

$$\begin{aligned}&\mathbf{f}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2, a_1\mathbf{u}_1 + a_2\mathbf{u}_2)\\ &= \mathbf{A}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + \mathbf{B}(a_1\mathbf{u}_1 + a_2\mathbf{u}_2)\\ &= a_1(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + a_2(\mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2)\\ &= a_1\mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) + a_2\mathbf{f}(\mathbf{x}_2, \mathbf{u}_2), \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{n_x}, \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^{n_u},\ \forall a_1, a_2 \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}&\mathbf{g}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2, a_1\mathbf{u}_1 + a_2\mathbf{u}_2)\\ &= \mathbf{C}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + \mathbf{D}(a_1\mathbf{u}_1 + a_2\mathbf{u}_2)\\ &= a_1(\mathbf{C}\mathbf{x}_1 + \mathbf{D}\mathbf{u}_1) + a_2(\mathbf{C}\mathbf{x}_2 + \mathbf{D}\mathbf{u}_2)\\ &= a_1\mathbf{g}(\mathbf{x}_1, \mathbf{u}_1) + a_2\mathbf{g}(\mathbf{x}_2, \mathbf{u}_2), \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{n_x}, \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^{n_u},\ \forall a_1, a_2 \in \mathbb{R}.\end{aligned}$$

Superposition and scaling hold!

- Consider the previous example; $\mathbf{f} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ can be partitioned as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\mathbf{B}} u(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$
$$y(t) = \underbrace{\begin{bmatrix} 0 & c \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{\mathbf{D}} u(t),$$

which is equivalent to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t).$$

Example

Q. Consider

$$\dddot{q}(t) + 2\ddot{q}(t) + q(t) = u(t), \qquad q(0) = q_0, \;\; \dot{q}(0) = \dot{q}_0, \;\; \ddot{q}(0) = \ddot{q}_0, \tag{3}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 3\dot{q}(t) + q(t) \\ \ddot{q}(t) \end{bmatrix}. \tag{4}$$

What kind of DE is in (3)? Write the system in state-space form. Is the system linear?

A. The DE in (3) is a third order, constant coefficient, nonhomogeneous, ODE.
Let

$$x_1(t) = q(t), \qquad x_2(t) = \dot{x}_1(t) = \dot{q}(t), \qquad x_3(t) = \dot{x}_2(t) = \ddot{x}_1(t) = \ddot{q}(t).$$

Then, write the ODE as

$$\begin{aligned} \dot{x}_3(t) + 2x_3(t) + x_1(t) &= u(t), \\ \dot{x}_3(t) &= -2x_3(t) - x_1(t) + u(t). \end{aligned}$$

Then,

$$\overbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}}^{\dot{\mathbf{x}}(t)} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}^{\mathbf{x}(t)} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{\mathbf{B}} u(t)$$

$$\mathbf{y}(t) = \underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{D}} u(t)$$

Thus, the system is linear. (More precisely, it's a linear time-invariant system.)

Normalization

- Consider the system

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t)), \qquad \tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0,$$
$$\tilde{\mathbf{y}}(t) = \mathbf{g}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{u}}(t)),$$

  where $\tilde{\mathbf{x}}(t)$, $\tilde{\mathbf{u}}(t)$, and $\tilde{\mathbf{y}}(t)$ all have units (e.g., m, $\mathsf{m/s}$, N, $\mathsf{N \cdot m}$, V etc.).

- Often systems are *normalized* using a change of variable resulting in all states, inputs, and outputs being unitless.

- Normalization boils down to

$$\tilde{x}_i(t) = x_{i,\text{nor}}\, x_i(t), \quad \tilde{u}_j(t) = u_{j,\text{nor}}\, u_j(t), \quad \tilde{y}_\ell(t) = y_{\ell,\text{nor}}\, y_\ell(t),$$

  where $x_i(t)$, $u_j(t)$, and $y_\ell(t)$ are unitless and $x_{i,\text{nor}}$, $u_{j,\text{nor}}$, and $y_{\ell,\text{nor}}$ are the normalization constants for each state, input, and output.

  - Often the $u_{j,\text{nor}}$'s are the largest (in absolute value) input values.
    - Say the j^{th} motor's max motor torque is $u_{\max} = 10$ ($\mathsf{N \cdot m}$). Then $u_{j,\text{nor}} = u_{\max}$.
  - Often the $y_{i,\text{nor}}$'s are the largest (in absolute value) measurement values.
    - Say the ℓ^{th} axis of a GPS receiver can measure a max velocity of $y_{\max} = 30$ ($\mathsf{m/s}$). Then $y_{\ell,\text{nor}} = y_{\max}$ for any measurement of velocity.

- In matrix form, normalization is

$$\tilde{\mathbf{x}}(t) = \mathbf{T}_x \mathbf{x}(t), \quad \tilde{\mathbf{u}}(t) = \mathbf{T}_u \mathbf{u}(t), \quad \tilde{\mathbf{y}}(t) = \mathbf{T}_y \mathbf{y}(t),$$

  where

$$\begin{aligned}
\mathbf{T}_x &= \mathrm{diag}(x_{1,\text{nor}}, \cdots, x_{n_x,\text{nor}}), \\
\mathbf{T}_u &= \mathrm{diag}(u_{1,\text{nor}}, \cdots, u_{n_u,\text{nor}}), \\
\mathbf{T}_y &= \mathrm{diag}(y_{1,\text{nor}}, \cdots, y_{n_y,\text{nor}}),
\end{aligned}$$

  and $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathbf{y}(t)$ are normalized (unitless).
- The normalized state-space system is then

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \overbrace{\mathbf{T}_x^{-1} \tilde{\mathbf{f}}(\mathbf{T}_x \mathbf{x}(t), \mathbf{T}_u \mathbf{u}(t))}^{\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))}, \qquad \mathbf{x}(0) = \mathbf{T}_x^{-1} \tilde{\mathbf{x}}(0), \\
\mathbf{y}(t) &= \underbrace{\mathbf{T}_y^{-1} \tilde{\mathbf{g}}(\mathbf{T}_x \mathbf{x}(t), \mathbf{T}_u \mathbf{u}(t))}_{\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t))}.
\end{aligned}$$

- Note that,
  - when computing $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{T}_x^{-1}\tilde{\mathbf{f}}(\mathbf{T}_x\mathbf{x}(t), \mathbf{T}_u\mathbf{u}(t))$, the units of each element of $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ will be s^{-1},
  - while when computing $\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{T}_y^{-1}\tilde{\mathbf{g}}(\mathbf{T}_x\mathbf{x}(t), \mathbf{T}_u\mathbf{u}(t))$, the units of each element of $\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$ will be truly unitless.
- $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ having units of s^{-1} is correct.
- To understand why, consider integrating

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) \in \mathbb{R}^{n_x},$$

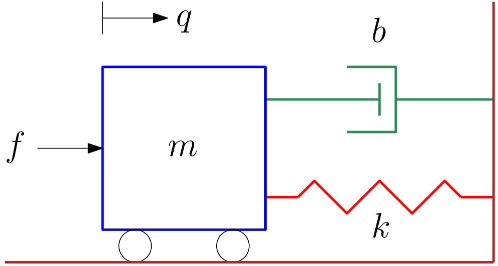
  resulting in

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau))\, \mathrm{d}\tau,$$

  where both $\mathbf{x}(0)$ and $\int_0^t \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau))\, \mathrm{d}\tau$ are unitless.
- When computing the integral $\int_0^t \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau))\, \mathrm{d}\tau$, the units are the product of the integrand units, that being the units of $\mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau))$, and the units of the infinitesimal element, that being the units of $\mathrm{d}\tau$.
  - Units of integrand = s^{-1}.
  - Units of infinitesimal element = s.
  - Units of integrand $\times$ Units of infinitesimal element = unitless.

- Generally computations are done in normalized form, while interpretation of computations are done in dimensional form.
- Be aware that the system may be, or may not be, normalized, and act accordingly.
- Sometimes it's convenient to work with units; other times it's convenient to work in non-dimensional form. Just be aware as to if or if not the system is normalized.

Example

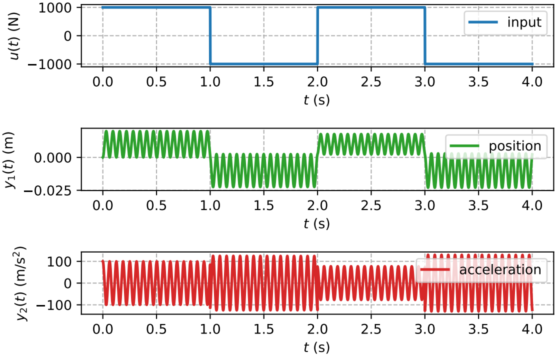


- Consider a mass-spring-damper where position $\tilde{y}_1(t) = q(t)$ is measured by a laser and acceleration $\tilde{y}_2(t) = \ddot{q}(t)$ is measured by an accelerometer.
- The state-space model is

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}^{\tilde{\mathbf{A}}} \overbrace{\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}}^{\tilde{\mathbf{x}}(t)} + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}^{\tilde{\mathbf{B}}} \tilde{u}(t),$$

$$\underbrace{\begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix}}_{\tilde{\mathbf{y}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{\tilde{\mathbf{C}}} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\tilde{\mathbf{D}}} \tilde{u}(t),$$

where $\tilde{x}_1(t) = q(t)$, $\tilde{x}_2(t) = \dot{q}(t)$, and $\tilde{u}(t) = f(t)$.



- The max (in absolute value) position is about 0.025 (m), and the max (in absolute value) velocity (which is not plotted) is about 1.5 (m/s). Thus,

$$\mathbf{T}_x = \mathrm{diag}(0.025, 1.5).$$

- The max (in absolute value) input is 1000 (N). Thus,

$$T_u = 1000.$$

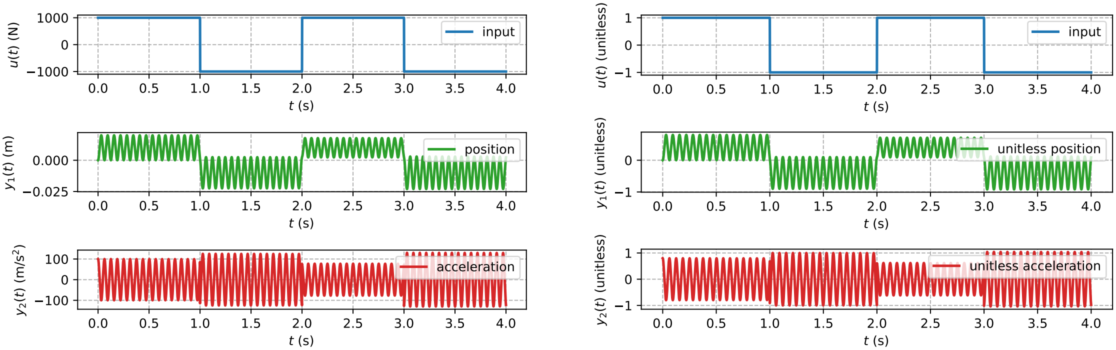
- The max (in absolute value) position measurement is about 0.025 (m), and the max (in absolute value) acceleration measurement is about 125 (m/s^2). Thus,

$$\mathbf{T}_y = \mathrm{diag}(0.025, 125).$$

- It follows that

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \overbrace{(\mathbf{T}_x^{-1}\tilde{\mathbf{A}}\mathbf{T}_x)}^{\mathbf{A}}\,\mathbf{x}(t) + \overbrace{(\mathbf{T}_x^{-1}\tilde{\mathbf{B}}T_u)}^{\mathbf{B}}\,u(t), \qquad \mathbf{x}(0) = \mathbf{T}_x^{-1}\tilde{\mathbf{x}}(0),\\ \mathbf{y}(t) &= \underbrace{(\mathbf{T}_y^{-1}\tilde{\mathbf{C}}\mathbf{T}_x)}_{\mathbf{C}}\,\mathbf{x}(t) + \underbrace{\mathbf{T}_y^{-1}\tilde{\mathbf{D}}T_u}_{\mathbf{D}}\,u(t).\end{aligned}$$

- When computing $\mathbf{A} = \mathbf{T}_x^{-1}\tilde{\mathbf{A}}\mathbf{T}_x$ and $\mathbf{B} = \mathbf{T}_x^{-1}\tilde{\mathbf{B}}T_u$, the units of each are s^{-1}, as expected. When computing $\mathbf{C} = \mathbf{T}_y^{-1}\tilde{\mathbf{C}}\mathbf{T}_x$ and $\mathbf{D} = \mathbf{T}_y^{-1}\tilde{\mathbf{D}}T_u$, the results are unitless, also as expected.



- The numerical values of each state-space model are

$$\left[\begin{array}{c|c} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \hline \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -10\,000 & -0.01 & 0.1 \\ \hline 1 & 0 & 0 \\ -10\,000 & -0.01 & 0.1 \end{array}\right],$$

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 60 & 0 \\ -166.67 & -0.01 & 66.67 \\ \hline 1 & 0 & 0 \\ -2 & -0.00012 & 0.8 \end{array}\right].$$

- Note that, when computing $\mathbf{A} = \mathbf{T}_x^{-1}\tilde{\mathbf{A}}\mathbf{T}_x$ and $\mathbf{B} = \mathbf{T}_x^{-1}\tilde{\mathbf{B}}T_u$, the units will be s^{-1}, while computing $\mathbf{C} = \mathbf{T}_y^{-1}\tilde{\mathbf{C}}\mathbf{T}_x$ and $\mathbf{D} = \mathbf{T}_y^{-1}\tilde{\mathbf{D}}T_u$, the result is unitless.

Linearization about an Equilibrium Point

- This class is primarily about analyzing linear plants and designing linear controllers. (But, using linear controllers we can still control nonlinear plants!)
- What do we do when confronted with a nonlinear system (usually a plant)? Linearize!
- Before discussing linearization, we must define an equilibrium point (EP). We will linearize about an EP.

Definition

Consider

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

An **equilibrium point** is any constant state and constant control input that satisfies

$$\mathbf{0} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}).$$

- What this means is that if $\mathbf{x}_0 = \bar{\mathbf{x}}$ and $\mathbf{u}(t) = \bar{\mathbf{u}}$, $\forall t \in \mathbb{R}^+$, the state remains at $\bar{\mathbf{x}}$ for all time.
- Note that the definition of an equilibrium point only involves the DE. The definition of an equilibrium point has nothing to do with the output equation $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$. However, using the equilibrium point the equilibrium output is defined as $\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$.

Linearization

- Now that we've defined an EP we can discuss linearization about an EP.
- To "linearize" means to perturb an EP and perform a Taylor-series expansion about the same EP.
- Consider the nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{5}$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)), \tag{6}$$

an EP, $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$, and perturbations to the EP such that

$$\mathbf{x}(t) = \bar{\mathbf{x}} + \delta\mathbf{x}(t), \tag{7}$$
$$\mathbf{u}(t) = \bar{\mathbf{u}} + \delta\mathbf{u}(t), \tag{8}$$
$$\mathbf{y}(t) = \bar{\mathbf{y}} + \delta\mathbf{y}(t). \tag{9}$$

- Substitution (7) and (8) into (5) and expanding $\mathbf{f}(\cdot,\cdot)$ using a Taylor-series expansion about $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ leads to

$$\begin{aligned}\underbrace{\dot{\bar{\mathbf{x}}}}_{=\mathbf{0}} + \ \delta\dot{\mathbf{x}}(t) &= \mathbf{f}(\bar{\mathbf{x}} + \delta\mathbf{x}(t), \bar{\mathbf{u}} + \delta\mathbf{u}(t)), \\ &= \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + \mathbf{A}\delta\mathbf{x}(t) + \mathbf{B}\delta\mathbf{u}(t) + \text{H.O.T.},\end{aligned} \tag{10}$$

where

$$\mathbf{A} \triangleq \left.\frac{\partial\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))}{\partial\mathbf{x}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, \mathbf{u}(t)=\bar{\mathbf{u}}},$$

$$\mathbf{B} \triangleq \left.\frac{\partial\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))}{\partial\mathbf{u}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, \mathbf{u}(t)=\bar{\mathbf{u}}}.$$

- Subtracting the nominal solution $\mathbf{0} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ from (10) and neglecting H.O.T. results in

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}\delta\mathbf{x}(t) + \mathbf{B}\delta\mathbf{u}(t).$$

- Substitution (7), (8), and (9) into (6) and expanding $\mathbf{g}(\cdot,\cdot)$ using a Taylor-series expansion about $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ leads to

$$\begin{aligned}\bar{\mathbf{y}} + \delta\mathbf{y}(t) &= \mathbf{g}(\bar{\mathbf{x}} + \delta\mathbf{x}(t), \bar{\mathbf{u}} + \delta\mathbf{u}(t)),\\ &= \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + \mathbf{C}\delta\mathbf{x}(t) + \mathbf{D}\delta\mathbf{u}(t) + \text{H.O.T.},\end{aligned} \tag{11}$$

where

$$\mathbf{C} \triangleq \left.\frac{\partial\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))}{\partial\mathbf{x}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, \mathbf{u}(t)=\bar{\mathbf{u}}},$$

$$\mathbf{D} \triangleq \left.\frac{\partial\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))}{\partial\mathbf{u}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, \mathbf{u}(t)=\bar{\mathbf{u}}}.$$

- Subtracting the nominal solution $\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ from (11) and neglecting H.O.T. results in

$$\delta\mathbf{y}(t) = \mathbf{C}\delta\mathbf{x}(t) + \mathbf{D}\delta\mathbf{u}(t).$$

- The linearized system is thus

$$\begin{aligned}\delta\dot{\mathbf{x}}(t) &= \mathbf{A}\delta\mathbf{x}(t) + \mathbf{B}\delta\mathbf{u}(t),\\ \delta\mathbf{y}(t) &= \mathbf{C}\delta\mathbf{x}(t) + \mathbf{D}\delta\mathbf{u}(t).\end{aligned}$$

- The linearized system is LTI because the state-space matrices are constant.

Example

Q. Find the EPs of

$$\ddot{q}(t) + (2 + \sin q(t))q(t) = u(t), \qquad q(0) = q_0, \;\; \dot{q}(0) = \dot{q}_0.$$

Verify that $\bar{q} = 0$, $\dot{\bar{q}} = 0$, $\bar{u} = 0$ is an EP. If

$$y(t) = (-2 + \cos q(t))q(t) + u(t),$$

write the system in first-order state-space form and linearize about $\bar{q} = 0$, $\dot{\bar{q}} = 0$, $\bar{u} = 0$.

A. Define the states to be

$$x_1(t) = q(t), \qquad x_2(t) = \dot{x}_1(t) = \dot{q}(t).$$

Then

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} x_2(t) \\ -(2 + \sin x_1(t))x_1(t) + u(t) \end{bmatrix}}_{\mathbf{f}(\mathbf{x}(t),u(t))}, \qquad x_1(0) = x_{1,0}, \;\; x_2(0) = x_{2,0}.$$

To find the EPs, find $\bar{\mathbf{x}}$ and $\bar{u}$ that satisfy

$$\mathbf{0} = \mathbf{f}(\bar{\mathbf{x}}, \bar{u}).$$

Thus, EPs are any $\bar{\mathbf{x}} = [\bar{x}_1 \quad \bar{x}_2]^{\mathsf{T}}$ and $\bar{u}$ such that

$$\begin{aligned} 0 &= \bar{x}_2, \\ 0 &= -(2 + \sin \bar{x}_1)\bar{x}_1 + \bar{u} \quad \Leftrightarrow \quad \bar{u} = (2 + \sin \bar{x}_1)\bar{x}_1. \end{aligned}$$

For example, valid EPs are

$$\begin{aligned} \bar{\mathbf{x}} &= [0 \quad 0]^{\mathsf{T}}, \quad \bar{u} = (2 + \sin 0)0 = 0, \\ \bar{\mathbf{x}} &= [\pi/2 \quad 0]^{\mathsf{T}}, \quad \bar{u} = (2 + \sin \pi/2)\pi/2 = 3\pi/2, \\ \bar{\mathbf{x}} &= [\pi \quad 0]^{\mathsf{T}}, \quad \bar{u} = (2 + \sin \pi)\pi = 2\pi, \\ \bar{\mathbf{x}} &= [3\pi/2 \quad 0]^{\mathsf{T}}, \quad \bar{u} = (2 + \sin 3\pi/2)3\pi/2 = 3\pi/2, \quad \text{etc.} \end{aligned}$$

Now, linearize about $\bar{\mathbf{x}} = [0 \quad 0]^\mathsf{T}$, $\bar{u} = 0$. The matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x}(t), u(t))}{\partial \mathbf{x}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, u(t)=\bar{u}} = \left.\begin{bmatrix} 0 & 1 \\ -(2+\sin x_1(t)) - \cos x_1(t) x_1(t) & 0 \end{bmatrix}\right|_{\bar{\mathbf{x}},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix},$$

$$\mathbf{B} = \left.\frac{\partial \mathbf{f}(\mathbf{x}(t), u(t))}{\partial u(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, u(t)=\bar{u}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To compute $\mathbf{C}$ and $\mathbf{D}$, first note that

$$y(t) = (-2 + \cos x_1(t)) x_1(t) + u(t).$$

Thus,

$$\mathbf{C} = \left.\frac{\partial \mathbf{g}(\mathbf{x}(t), u(t))}{\partial \mathbf{x}(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, u(t)=\bar{u}} = \left.\begin{bmatrix} (-2+\cos x_1(t)) + (-\sin x_1(t)) x_1 & 0 \end{bmatrix}\right|_{\bar{\mathbf{x}},\bar{u}} = \begin{bmatrix} -1 & 0 \end{bmatrix},$$

$$\mathbf{D} = \left.\frac{\partial \mathbf{g}(\mathbf{x}(t), u(t))}{\partial u(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}, u(t)=\bar{u}} = 1.$$

It follows that the linearized system is

$$
\begin{aligned}
\delta\dot{\mathbf{x}}(t) &= \mathbf{A}\delta\mathbf{x}(t) + \mathbf{B}\delta u(t), \\
\delta y(t) &= \mathbf{C}\delta\mathbf{x}(t) + \mathbf{D}\delta u(t),
\end{aligned}
$$

or, more explicitly,

$$
\begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u(t)
$$

$$
\delta y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} + \delta u(t).
$$

Appendix - Jacobians

- Consider the general function $\mathbf{f}(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_{m-1}(\mathbf{x}) \\ f_m(\mathbf{x}) \end{bmatrix}. \tag{12}$$

- Using a Taylor series expansion the first-order approximation of $\mathbf{f}(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is

$$\mathbf{f}(\bar{\mathbf{x}} + \delta\mathbf{x}) \approx \mathbf{f}(\bar{\mathbf{x}}) + \underbrace{\left[\left. \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\bar{\mathbf{x}}} \right] \delta\mathbf{x}}_{\delta\mathbf{f}(\bar{\mathbf{x}})} \tag{13}$$

where the Jacobian is

$$\left. \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left. \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_{n-1}} & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_{n-1}} & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_m(\mathbf{x})}{\partial x_1} & \frac{\partial f_m(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_m(\mathbf{x})}{\partial x_{n-1}} & \frac{\partial f_m(\mathbf{x})}{\partial x_n} \end{bmatrix} \right|_{\mathbf{x}=\bar{\mathbf{x}}}. \tag{14}$$

Example

- Consider

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 3x_1 \\ -x_1 x_2^2 \\ x_1 + x_2 \end{bmatrix}, \tag{15}$$

  where $\mathbf{x} = [x_1 \quad x_2]^\mathsf{T}$.

- Using a Taylor series expansion it can be shown that the Jacobian is

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 3 & 0 \\ -x_2^2 & -2x_1 x_2 \\ 1 & 1 \end{bmatrix}. \tag{16}$$

- Can also compute (16) directly “by hand” (i.e., without the Taylor series).

Notes and References

Notes
See Section 2.2 of [1], Section 3.2 of [2], and Chapter 1 of [3] for Jacobian computation.

References

[1] L. Qiu and K. Zhou, *Introduction to Feedback Control*. Upper Saddle River, NJ: Prentice-Hall, Inc., 2010.
[2] L. Guzzella, *Analysis and Synthesis of Single-Input Single-Output Control Systems*, Fourth. ETH Zurich: vdf Hochschulverlag AG, 2019.
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