Special Functions and Laplace Transforms
MECH 412 - *System Dynamics and Control*

Prof. James Richard Forbes ©

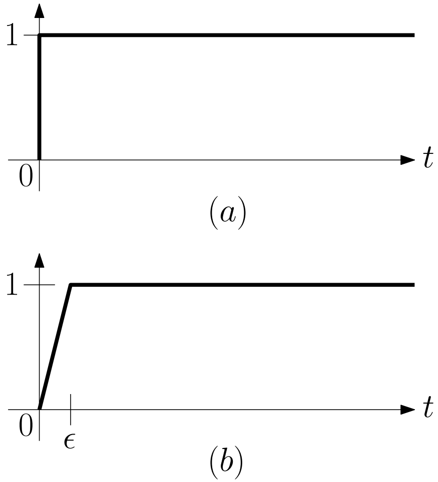
McGill University, Department of Mechanical Engineering

Unit Step

- A *unit step* applied at $t = 0$ is defined as

$$1_+(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} .$$

- Referring to Figure (b), as $\epsilon \to 0$ the function $1_+(t)$ is recovered.



$1_+(t)$ is denoted $\sigma(t)$ in [1].

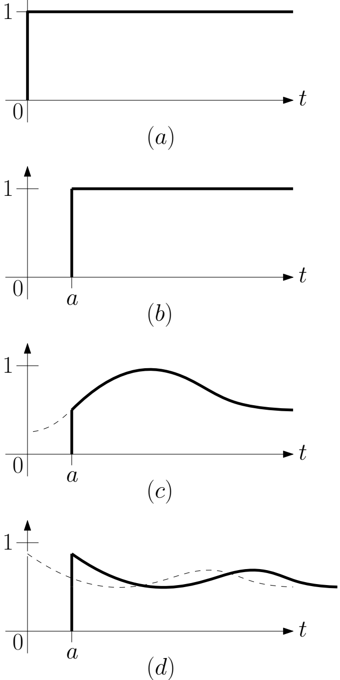
(a) $1_+(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$

(b) $1_+(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}.$

(c) $1_+(t-a)f(t) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}.$

(d) $1_+(t-a)f(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}.$

Note, $0 \leq a < \infty$.



Unit Impulse (Delta Function)

- A *unit impulse* applied at $t = 0$ is defined as

$$\begin{cases} \delta(t) = 0, \quad t \neq 0 \\ \int_0^t \delta(\tau)\mathrm{d}\tau = 1_+(t) \end{cases},$$

while an equivalent definition of $\delta(t)$ is

$$\delta(t) = \frac{\mathrm{d}1_+(t)}{\mathrm{d}t}.$$

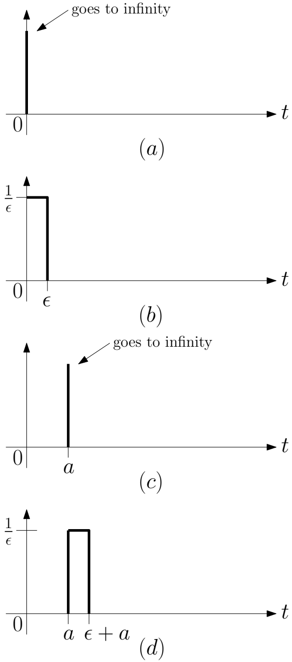
- A *unit impulse* applied at $t = a$, where $0 \leq a < \infty$, is defined as

$$\begin{cases} \delta(t-a) = 0, \quad t \neq a \\ \int_0^t \delta(\tau - a)\mathrm{d}\tau = 1_+(t-a) \end{cases},$$

while an equivalent definition of $\delta(t-a)$ is given by

$$\delta(t-a) = \frac{\mathrm{d}1_+(t-a)}{\mathrm{d}t}.$$

- The unit impulse $\delta(t-a)$ satisfies $\int_0^t \delta(\tau - a)f(\tau)\mathrm{d}\tau = f(a)$.



Laplace Transforms

- The general idea of a transform is to re-express the time-domain function in the s-domain, that is
$$\mathcal{L}\{f(t)\} = f(s), \qquad \text{where} \qquad f(s) = \int_a^b \kappa(s,t) f(t) \mathrm{d}t.$$
- Using the kernal function
$$\kappa(s,t) = e^{-st}$$
and $a = 0$ and $b = \infty$, where $s = \operatorname{Re} s + \jmath \operatorname{Im} s = \sigma + \jmath\omega$ leads to the Laplace transform (LT)
$$\mathcal{L}\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$
- $\{f(t), f(s)\}$ is a LT pair.
- Why use the LT transform at all?
  - Can simplify a problem. Hard problem with $f(t)$ but easy problem with $f(s)$.
  - Can convert an ODE into an into an algebraic equation.
  - Easier to handle discontinuities.

Assumptions

- Only functions of the form $f(t) = 0$, $\forall t < 0$, will be considered.
- Sufficient conditions for an LT pair to exist are as follows.
  - $f(t)$, $\forall t \geq 0$, is piecewise continous.
  - $f(t)$, $\forall t \geq 0$, "grows" no faster than than $\alpha e^{\beta t}$, $\forall t \geq 0$, where $\beta \in \mathbb{R}$.
  - For example, $f(t) = e^{t^2} 1_+(t)$ does not have an LT becuase it grows faster than $\alpha e^{\beta t} 1_+(t)$ for any β.

The Laplace Transform is a Linear* Operator

$$
\begin{aligned}
\mathcal{L}\left\{a_1 f_1(t) + a_2 f_2(t)\right\} &= \int_0^\infty e^{-st}\left(a_1 f_1(t) + a_2 f_2(t)\right)\mathrm{d}t \\
&= \int_0^\infty e^{-st}\left(a_1 f_1(t)\right)\mathrm{d}t + \int_0^\infty e^{-st}\left(a_2 f_2(t)\right)\mathrm{d}t \\
&= a_1 \int_0^\infty e^{-st} f_1(t)\mathrm{d}t + a_2 \int_0^\infty e^{-st} f_2(t)\mathrm{d}t \\
&= a_1 \mathcal{L}\left\{f_1(t)\right\} + a_2 \mathcal{L}\left\{f_2(t)\right\}
\end{aligned}
$$

where $\operatorname{Re} s > \max(\sigma_1, \sigma_2)$ and σ_1 and σ_2 define the ROC of the LTs $f_1(s)$ and $f_2(s)$

*Recall, “linear” means and hold.

Inverse Lapace Transform

- The inverse LT is defined to be

$$\mathcal{L}^{-1}\left\{f(s)\right\} = f(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2\pi \jmath}\int_{\sigma_c - \jmath\infty}^{\sigma_c + \jmath\infty} e^{st} f(s)\mathrm{d}s, & t \geq 0 \end{cases},$$

  where σ_c defines the ROC. Turns out, this definition is not all that useful in practice.

- To find the inverse Laplace transform, use LT tables,

$$f(t) \xrightarrow{\text{LT table}} f(s),$$

$$f(s) \xrightarrow{\text{LT table}} f(t),$$

  where $\mathcal{L}\left\{f(t)\right\} = f(s)$ and $\mathcal{L}^{-1}\left\{f(s)\right\} = f(t)$.

- The inverse LT is also a linear operator, that is

$$\begin{aligned} \mathcal{L}^{-1}\left\{a_1 f_1(s) + a_2 f_2(s)\right\} &= a_1 \mathcal{L}^{-1}\left\{f_1(s)\right\} + a_2 \mathcal{L}^{-1}\left\{f_2(s)\right\} \\ &= a_1 f_1(t) + a_2 f_2(t). \end{aligned}$$

Common LT Pairs

$f(s) = \int_0^\infty e^{-st} f(t)dt$	$f(t),\ \forall t \geq 0$
1	$\delta(t)$
$\frac{1}{s}$	$1_+(t)$
$\frac{1}{s^2}$	t
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{a}{s(s+a)}$	$1 - e^{-at}$
$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$
$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos(bt)$
$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin(bt)$

Example

Q. What is the LT of $f(t) = 1_+(t-a)$ where $0 \le a < \infty$?

A.

$$\begin{aligned}
\mathcal{L}\{1_+(t-a)\} &= \int_0^\infty e^{-st} 1_+(t-a)\mathrm{d}t \\
&= \int_a^\infty e^{-st}\mathrm{d}t \\
&= -\frac{1}{s}\lim_{t\to\infty} e^{-st} - \left(-\frac{1}{s}\right) e^{-sa} \\
&= \frac{1}{s}e^{-sa},
\end{aligned}$$

when $\operatorname{Re} s > 0$.

Q. What is the LT of $f(t) = 1_+(t)$?

Example

Q. What is the LT of $\delta(t-a)$ where $0 \leq a < \infty$?

A. Recall that

$$\int_0^t \delta(\tau - a) f(\tau) \mathrm{d}\tau = f(a).$$

Thus

$$\begin{aligned} \mathcal{L}\left\{\delta(t-a)\right\} &= \int_0^\infty e^{-st}\delta(t-a)\mathrm{d}t \\ &= e^{-sa}. \end{aligned}$$

Example

Q. What's the LT of $f(t) = e^{kt}1_+(t)$ where $k \in \mathbb{C}$?

A.

$$\begin{aligned}
\mathcal{L}\left\{e^{kt}1_+(t)\right\} &= \int_0^\infty e^{-st}e^{kt}1_+(t)\mathrm{d}t \\
&= \int_0^\infty e^{-st}e^{kt}\mathrm{d}t \\
&= \int_0^\infty e^{(k-s)t}\mathrm{d}t \\
&= \frac{1}{(k-s)}\lim_{t\to\infty} e^{-(s-k)t} - \frac{1}{(k-s)}\lim_{t\to 0} e^{-(s-k)t}
\end{aligned}$$

Let $s = \sigma + \jmath\omega$ and $k = u + \jmath v$ so that

$$s - k = (\sigma - u) + \jmath(\omega - v) = \alpha + \jmath\beta.$$

Recall Euler's formula, $e^{\jmath at} = \cos at + \jmath \sin at$.

Thus,

$$\begin{aligned}
e^{-(s-k)t} &= e^{-(\alpha+\jmath\beta)t} \\
&= e^{-\alpha t}e^{\jmath(-\beta)t} \\
&= e^{-\alpha t}(\cos(-\beta)t + \jmath\sin(-\beta)t) \\
&= e^{-\alpha t}(\cos\beta t - \jmath\sin\beta t)
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{L}\left\{e^{kt}1_{+}(t)\right\} &= \frac{1}{(k-s)}\lim_{t\to\infty}\left(e^{-\alpha t}(\cos(\beta t) - \jmath\sin(\beta t))\right) + \frac{1}{(s-k)} \\
&= \frac{1}{(s-k)}
\end{aligned}$$

when $\operatorname{Re} s > \operatorname{Re} k$.

Example

Q. What's the LT of $f_c(t) = \cos(at)1_+(t)$ and $f_s(t) = \sin(at)1_+(t)$ where $a \in \mathbb{R}$?

A. From Euler's Formula,

$$e^{\jmath at} = \cos(at) + \jmath \sin(at).$$

Also recall that $\mathcal{L}\left\{e^{kt}1_+(t)\right\} = \frac{1}{s-k}$ where $k \in \mathbb{C}$. Thus,

$$\begin{aligned}
\mathcal{L}\left\{e^{\jmath at}1_+(t)\right\} &= \frac{1}{(s-\jmath a)} \frac{s+\jmath a}{s+\jmath a} \\
&= \frac{s}{(s^2+a^2)} + \jmath \frac{a}{(s^2+a^2)},
\end{aligned}$$

when $\operatorname{Re} s > \operatorname{Re} k$, meaning $\operatorname{Re} s > 0$ because $k = 0 + \jmath a$ in this case. Next, note that

$$\begin{aligned}
\mathcal{L}\left\{e^{\jmath at}1_+(t)\right\} &= \int_0^\infty e^{-st}\left(\cos(at) + \jmath \sin(at)\right)1_+(t)\mathrm{d}t \\
&= \int_0^\infty e^{-st}\cos(at)1_+(t)\mathrm{d}t + \jmath \int_0^\infty e^{-st}\sin(at)1_+(t)\mathrm{d}t \\
&= \mathcal{L}\left\{f_c(t)\right\} + \jmath \mathcal{L}\left\{f_s(t)\right\}.
\end{aligned}$$

Thus,

$$\mathcal{L}\left\{f_c(t)\right\} = \mathcal{L}\left\{\cos(at)1_+(t)\right\} = \frac{s}{(s^2+a^2)}, \qquad \mathcal{L}\left\{f_s(t)\right\} = \mathcal{L}\left\{\sin(at)1_+(t)\right\} = \frac{a}{(s^2+a^2)}.$$

Differentiation

Claim: $\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sf(s) - f(0)$ where $f(0) = f(t)|_{t=0}$.

Proof.

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = \int_0^\infty e^{-st}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t = \int_0^\infty e^{-st}\mathrm{d}f(t)$$

Use integration by parts,

$$\begin{aligned} u(t) = e^{-st}, &\quad \mathrm{d}v(t) = \mathrm{d}f(t), \\ \mathrm{d}u(t) = -se^{-st}\mathrm{d}t, &\quad v(t) = f(t), \\ \int_a^b u(t)\mathrm{d}v(t) &= u(t)v(t)|_a^b - \int_a^b v(t)\mathrm{d}u(t). \end{aligned}$$

With $\operatorname{Re} s > \operatorname{Re} \sigma$ where σ defines the ROC of the LT of $f(t)$, it follows that

$$\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} &= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\mathrm{d}t \\
&= s\int_0^\infty f(t)e^{-st}\mathrm{d}t - f(0) \\
&= sf(s) - f(0)
\end{aligned}$$

In general,

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = s^n f(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{n-1}(0),$$

where $f^{n-1}(t)$ is the $n-1^{th}$ derivative.

Integration

Claim: Let $f_1(t) = \int_0^t f(\tau)\mathrm{d}\tau$. Then, $\mathcal{L}\left\{f_1(t)\right\} = \frac{1}{s}f(s)$.

Proof.

$$\mathcal{L}\left\{f_1(t)\right\} = \int_0^\infty e^{-st} f_1(t)\mathrm{d}t = \int_0^\infty e^{-st} \int_0^t f(\tau)\mathrm{d}\tau\mathrm{d}t$$

Use integration by parts,

$$u(t) = \int_0^t f(\tau)\mathrm{d}\tau, \quad \mathrm{d}v(t) = e^{-st}\mathrm{d}t,$$

$$\mathrm{d}u(t) = f(t)\mathrm{d}t, \quad v(t) = -\frac{1}{s}e^{-st},$$

$$\int_a^b u(t)\mathrm{d}v(t) = u(t)v(t)|_a^b - \int_a^b v(t)\mathrm{d}u(t).$$

it follows that

$$\mathcal{L}\left\{f_1(t)\right\} = \underbrace{\left(\int_0^t f(\tau)\mathrm{d}\tau\right)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty}_{\text{goes to zero}} - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right) f(t)\mathrm{d}t$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}\mathrm{d}t = \frac{1}{s}\mathcal{L}\left\{f(t)\right\} = \frac{1}{s}f(s)$$

Convolution

- Does $\mathcal{L}\left\{f(t)g(t)\right\} \stackrel{?}{=} f(s)g(s)$?

No!

- The convolution of $f(t)$ and $g(t)$ is

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau, \qquad \forall t \geq 0.$$

- The LT of the convolution of $f(t)$ and $g(t)$ is $\mathcal{L}\left\{f(t) * g(t)\right\} = f(s)g(s)$.

Example

Q. Given $f(s) = \frac{1}{(s+1)(s+2)}$ what is $f(t)$, $\forall t \geq 0$?

A. Consider

$$f_1(s) = \frac{1}{(s+1)} \quad \text{and} \quad f_2(s) = \frac{1}{(s+2)}$$

$$f_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} = e^{-t}1_+(t) \quad \text{and} \quad f_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} = e^{-2t}1_+(t)$$

It follows that

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{f_1(s)f_2(s)\} &= \int_0^t f_1(t-\tau)f_2(\tau)\mathrm{d}\tau \\ &= \int_0^t e^{-(t-\tau)}1_+(t-\tau)e^{-2\tau}1_+(\tau)\mathrm{d}\tau \\ &= e^{-t}\int_0^t e^{-\tau}1_+(t-\tau)\mathrm{d}\tau, \quad \forall t \geq 0. \end{aligned}$$

Letting $\eta = t - \tau$, noting that $\mathrm{d}\eta = -\mathrm{d}\tau$, and changing the integration bounds gives

$$\begin{aligned} f(t) = e^{-t}\int_t^0 e^{-(t-\eta)}1_+(\eta)(-\mathrm{d}\eta) = e^{-t}\int_0^t e^{-(t-\eta)}\mathrm{d}\eta = e^{-2t}\int_0^t e^{\eta}\mathrm{d}\eta &= e^{-2t}(e^t - 1) \\ &= e^{-t} - e^{-2t}, \quad \forall t \geq 0. \end{aligned}$$

Partial-Fraction Expansion

- What is $f(t) = \mathcal{L}^{-1}\{f(s)\}$ where $f(s) = \frac{1}{(s+3)(s-2)}$?
- Could use convolution ... but that's a lot of work. Is there an easier way?
- **Yes!** Partial fraction expansion!
- First recall that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}1_+(t), \quad \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}1_+(t).$$

- Write

$$\frac{1}{(s+3)(s-2)} = \frac{c_1}{s+3} + \frac{c_2}{s-2}$$

and then solve for c_1 and c_2.

- Rearranging,

$$
\begin{aligned}
\frac{1}{(s+3)(s-2)} &= \frac{c_1}{s+3} + \frac{c_2}{s-2}, \\
1 &= (s+3)(s-2)\left(\frac{c_1}{s+3} + \frac{c_2}{s-2}\right) \\
&= (s-2)c_1 + (s+3)c_2, \\
(0)s + (1)s^0 &= (c_1 + c_2)s + (-2c_1 + 3c_2)s^0.
\end{aligned}
$$

- We have two equations, and two unknowns.
- The first equation comes from s^0,

$$1 = -2c_1 + 3c_2 = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

- The second comes from $s^1 = s$,

$$0 = c_1 + c_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

- Writing these two equations in matrix form gives

$$\begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- This is a classic $\mathbf{Ax} = \mathbf{b}$ problem.
- Recalling that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

  it follows that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{1}{5} \end{bmatrix}.$$

- Find that $c_1 = -\frac{1}{5}$ and $c_2 = \frac{1}{5}$, and therefore

$$f(s) = \frac{1}{(s+3)(s-2)} = \frac{-\frac{1}{5}}{s+3} + \frac{\frac{1}{5}}{s-2}.$$

- It follows that

$$f(t) = \mathcal{L}^{-1}\left\{f(s)\right\} = \left(-\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t}\right)1_+(t).$$

- More examples can be found in Appendix C.3 of

`https://github.com/randybeard/controlbook_public`

Final-Value Theorem (Important!)

Let $f(s) = \mathcal{L}\{f(t)\}$ and assume that $sf(s)$ is analytic in $\operatorname{Re} s \geq 0$ (that is, $sf(s)$ has no poles in the CRHP). Then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s).$$

Example

Q. Compute $\lim_{t\to\infty} f(t)$ where $f(s) = \frac{1}{s(s+1)}$.

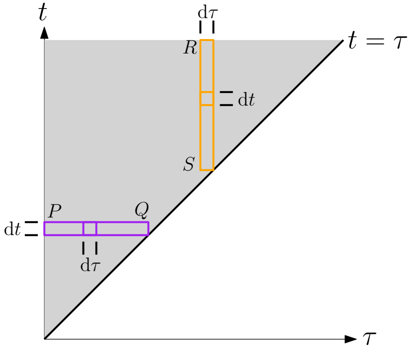
A. Note that $sf(s) = \frac{1}{s+1}$ has no poles in the CRHP; said another way, all poles of $sf(s) = \frac{1}{s+1}$ are in the OLHP. Thus, the FVT can be applied, yielding

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sf(s) \\ &= \lim_{s\to 0} \frac{1}{s+1} \\ &= 1.\end{aligned}$$

Appendix - Proof of Convolution Relation

- To show that $\mathcal{L}\{f(t) * g(t)\} = f(s)g(s)$, first consider

$$\begin{aligned}\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}t \\ &= \int_0^\infty \int_0^t f(t-\tau)g(\tau)e^{-st}\mathrm{d}\tau\mathrm{d}t\end{aligned}$$



- In the original form of the double integral τ goes from $\tau = 0$ to $\tau = t$ along PQ, and t goes from $t = 0$ to $t = \infty$.
- When the order of integration is changed, t goes from $t = \tau$ to $t = \infty$ along RS, and τ goes from $\tau = 0$ to $\tau = \infty$.
- Thus,

$$\mathcal{L}\left\{f(t) * g(t)\right\} = \int_0^\infty \int_0^t f(t-\tau)g(\tau)e^{-st}\mathrm{d}\tau\mathrm{d}t = \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)e^{-st}\mathrm{d}t\mathrm{d}\tau.$$

- Within the inner integral (on t), apply a change of variables. Specifically, $\eta = t - \tau$ (or $t = \eta + \tau$).
- t goes from $t = \tau$ to $t = \infty$, so η goes from $\eta = 0$ to $\eta = \infty$. Also, $\mathrm{d}\eta = \mathrm{d}t$.
- Therefore,

$$
\begin{aligned}
\mathcal{L}\left\{f(t) * g(t)\right\} &= \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)e^{-st}\mathrm{d}t\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty f(\eta)g(\tau)e^{-s(\eta+\tau)}\mathrm{d}\eta\mathrm{d}\tau \\
&= \int_0^\infty g(\tau)\left(\int_0^\infty f(\eta)e^{-s\eta}\mathrm{d}\eta\right)e^{-s\tau}\mathrm{d}\tau \\
&= \int_0^\infty g(\tau)f(s)e^{-s\tau}\mathrm{d}\tau \\
&= f(s)\int_0^\infty g(\tau)e^{-s\tau}\mathrm{d}\tau \\
&= f(s)g(s).
\end{aligned}
$$

Appendix - Proof of the Final-Value Theorem

To show that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s),$$

consider

$$\begin{aligned}
\mathcal{L}\left\{\dot{f}(t)\right\} &= \int_0^\infty e^{-st}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t,\\
\lim_{s\to 0}\mathcal{L}\left\{\dot{f}(t)\right\} &= \lim_{s\to 0}\int_0^\infty e^{-st}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t,\\
\lim_{s\to 0}\left(sf(s) - f(0)\right), &= \int_0^\infty \lim_{s\to 0} e^{-st}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t\\
\lim_{s\to 0} sf(s) - f(0) &= \int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t\\
&= \int_0^\infty \mathrm{d}f(t)\\
&= \lim_{t\to\infty} f(t) - f(0).
\end{aligned}$$

Thus,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s).$$

Notes and References

Notes
See Appendix A and Sections 1.2 of [1]. Step function and impulse function pictures from [1]. See [2] Chapter 2 as well.

Additionally, these notes are partially based on / inspired by notes by Prof. M. Peet, ASU, Prof. C.J. Damaren, UTIAS, Prof. T.D. Barfoot, UTAIS, and Prof. J. Hoagg, U. Kentucky.

[1] L. Qiu and K. Zhou, *Introduction to Feedback Control*. Upper Saddle River, NJ: Prentice-Hall, Inc., 2010.
[2] W. J. Palm, *System Dynamics*, 3rd. Toronto, ON: McGraw Hill, 2013.