Transfer Functions and ODE Solutions
MECH 412 - *System Dynamics and Control*

Prof. James Richard Forbes ©

McGill University, Department of Mechanical Engineering

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- Recall

$$\mathcal{L}\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t,$$
$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sf(s) - f(0),$$
$$\mathcal{L}\left\{\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right\} = s^2 f(s) - sf(0) - \dot{f}(0),$$
$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = s^n f(s) - s^{n-1} f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{n-1}(0)$$

  where $f(0) = f(t)|_{t=0}$, $\dot{f}(0) = \dot{f}(t)|_{t=0}$, and $f^{n-1}(t)$ is the $n-1^{th}$ derivative.
- In general,
  - differentiation with respect to time means multiply by s, and
  - integration with respect to time means divide by s.
- LTs can be used to find the solution to an ODE.

Example

Q. Consider the ODE

$$\dot{q}(t) + aq(t) = u(t), \quad \forall t \geq 0, \quad q(0) = q_0.$$

Find the solution $q(t)$, $\forall t \geq 0$, for **a)** a generic input $u(t)$, **b)** for the specific input $u(t) = m1_+(t)$, and **c)** for the specific input $u(t) = \delta(t)$.

A. Multiply both sides of the ODE by e^{-st} and then integrate both sides between $t = 0$ and $t = \infty$ to obtain

$$\int_0^\infty e^{-st}\left(\dot{q}(t) + aq(t)\right)\mathrm{d}t = \int_0^\infty e^{-st}u(t)\mathrm{d}t,$$

$$\int_0^\infty e^{-st}\dot{q}(t) + a\int_0^\infty e^{-st}q(t)\mathrm{d}t = \int_0^\infty e^{-st}u(t)\mathrm{d}t,$$

$$\mathcal{L}\left\{\dot{q}(t)\right\} + a\mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{u(t)\right\},$$

$$sq(s) - q_0 + aq(s) = u(s),$$

$$(s+a)q(s) = u(s) + q_0,$$

$$q(s) = \underbrace{\left(\frac{1}{s+a}\right)u(s)}_{\text{forced response}} + \underbrace{\frac{q_0}{s+a}}_{\text{free response}}$$

Thus, **a)** given a generic input $u(t)$ the time-domain solution to the ODE is

$$q(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s+a}\right)u(s)\right\} + \mathcal{L}^{-1}\left\{\frac{q_0}{s+a}\right\}$$
$$= \int_0^t e^{-a(t-\tau)}u(\tau)\mathrm{d}\tau + q_0 e^{-at}, \quad \forall t \geq 0.$$

Recall that $m\mathcal{L}\left\{1_+(t)\right\} = \frac{m}{s}$. Thus,

$$q(s) = \left(\frac{1}{s+a}\right)\frac{m}{s} + \frac{q_0}{s+a}.$$

Apply partial fraction expansion (PFE) to the first term. To begin,

$$\left(\frac{1}{s+a}\right)\frac{m}{s} = \frac{c_1}{s} + \frac{c_2}{s+a}.$$

It can be shown that $c_1 = \frac{m}{a}$ and $c_2 = -\frac{m}{a}$. As such,

$$q(s) = \frac{\frac{m}{a}}{s} + \frac{-\frac{m}{a}}{s+a} + \frac{q_0}{s+a}$$
$$= \frac{\frac{m}{a}}{s} + \frac{1}{s+a}\left(q_0 - \frac{m}{a}\right)$$

Thus, **b)** the time-domain solution to the ODE given the specific input $u(t) = m1_+(t)$ is

$$q(t) = \left[\frac{m}{a} + \left(q_0 - \frac{m}{a}\right)e^{-at}\right]1_+(t)$$

Recall that $\mathcal{L}\{\delta(t)\} = 1$. It follows that

$$
\begin{aligned}
q(s) &= \frac{1}{s+a} + \frac{q_0}{s+a} \\
&= \frac{1+q_0}{s+a}.
\end{aligned}
$$

Thus, **c)** the time-domain solution to the ODE given the specific input $u(t) = \delta(t)$ is

$$q(t) = (1+q_0)e^{-at}1_+(t).$$

Discussion

- Consider the previous example where $u(t) = \delta(t)$.

Q. When $0 < a < \infty$ does $\lim_{t\to\infty} q(t) = 0$, $\lim_{t\to\infty} q(t) = c$, or $\lim_{t\to\infty} q(t) = \infty$?

A. The solution is

$$q(t) = (1 + q_0)e^{-at}1_+(t).$$

- When $0 < a < \infty$, $\lim_{t\to\infty} q(t) = 0$.
- When $a = 0$, $\lim_{t\to\infty} q(t) = c$, where $c = 1 + q_0$.
- When $-\infty < a < 0$, $\lim_{t\to\infty} q(t) = \infty$.

Discussion Continued

- In most (almost all) engineering situations, $\lim_{t\to\infty} q(t) = \infty$ means something "blowed up" …which is bad!
- A major theme of this course is how to make any system achieve $\lim_{t\to\infty} q(t) = 0$[1] even in the presence of
  - model uncertainty (i.e., we don't know a exactly nor do we know q_0 at all),
  - disturbances,
  - noise.

[1] Or, in a tracking problem, $\lim_{t\to\infty} e(t) = 0$ where $e(t) = r(t) - q(t)$ and $r(t)$ is a reference, or command, to be tracked.

Transfer Functions

- Once again consider the ODE

$$\dot{q}(t) + aq(t) = u(t), \quad \forall t \geq 0, \quad q(0) = q_0.$$

- A sensor measures $q(t)$, thus $y(t) = q(t)$.
- The relationship between the input $u(t)$, the output (measurement) $y(t)$, and the IC q_0 is given by

$$y(s) = \left(\frac{1}{s+a}\right) u(s) + \frac{q_0}{s+a},$$

  where $G(s) = \frac{1}{s+a}$ is the system **transfer function**.
  - When we want to be extra specific about the transfer function inputs and outputs write $G_{yu}(s)$.
  - In doing so, notice that the "subscripts line up": $y(s) = G_{yu}(s)u(s)$.
- When ICs are quiescent the system transfer function can be written as

$$G(s) = \frac{y(s)}{u(s)}.$$

- The system transfer function encodes a lot of very useful information about the system (e.g., notions of "stability", notions of "hard to control" vs. "easy to control", etc.)

Transfer Functions

- Transfer functions are complex functions that are (usually) the ratio of two polynomials,

$$G(s) = \frac{b(s)}{a(s)}.$$

  - A transfer function will have the form $G(s) = \frac{b(s)}{a(s)}e^{-Ts}$ when the input is delayed by time $0 \le T < \infty$.

- It is assumed that transfer functions are **coprime**, meaning $b(s)$ and $a(s)$ do not have common factors. For example,
  - $b(s) = s - 1$ and $a(s) = s + 1$ are coprime, while
  - $b(s) = s - 1$ and $a(s) = (s+1)(s-1)$ are not coprime.

- A transfer function $G : \mathbb{C} \to \mathbb{C}$ can be written in rectangular form,

$$G(s) = \mathrm{Re}\left\{G(s)\right\} + \jmath\,\mathrm{Im}\left\{G(s)\right\},$$

  or in polar form,

$$G(s) = |G(s)|e^{\jmath\theta},$$

  where

$$|G(s)| = \sqrt{\mathrm{Re}\left\{G(s)\right\}^2 + \mathrm{Im}\left\{G(s)\right\}^2}, \quad \tan\theta = \frac{\mathrm{Im}\left\{G(s)\right\}}{\mathrm{Re}\left\{G(s)\right\}}.$$

Zeros of a Transfer Function

- Consider a rational transfer function,

$$G(s) = \frac{b(s)}{a(s)}.$$

- The roots of $b(s)$ (the numerator polynomial) are called the **zeros** of the transfer function.
- For example, the transfer function

$$G(s) = \frac{s+2}{(s+3)(s+4)^2}$$

  has a zero at

$$s = -2.$$

Poles of a Transfer Function

- Consider a rational transfer function,

$$G(s) = \frac{b(s)}{a(s)}.$$

- The roots of $a(s)$ (the denominator polynomial) are called the **poles** of the transfer function.
- Also, p_i is a pole of order n_i if

$$\lim_{s \to p_i} (s - p_i)^{n_i} G(s) = c$$

  where c is a constant. If $n_i = 1$ the pole is a simple pole.
- For example, the transfer function

$$G(s) = \frac{s+2}{(s+3)(s+4)^2}$$

  has a simple pole at

$$s = -3$$

  and pole of order 2 at

$$s = -4.$$

Relative Degree and Properness

- Consider a rational transfer function,

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
$$= k\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}.$$

- The value of $G(0)$ is the DC-gain of $G(s)$.
- The parameter k is the high-frequency gain of $G(s)$.
- The relative degree of the transfer function is $d = n - m$.
- If
  - $n \geq m$ the TF is proper;
  - $n > m$ the TF is strictly proper;
  - $n = m$ the TF is bi-proper;
  - $n < m$ the TF is improper.
- Real systems (i.e., mechanical systems, electrical systems, etc.) are never improper, and bi-proper TFs are less common than strictly proper TFs.

Impulse Response

- Consider $y(s) = G(s)u(s)$ where $u(t) = \delta(t)$. Then,
$$y(s) = G(s)u(s) = G(s),$$
which follows from the fact that $u(s) = \mathcal{L}\{\delta(t)\} = 1$.
- Taking the inverse LT of both sides yields
$$y(t) = G(t)1_+(t),$$
which is the impulse response.
- The same result can be attained directly from the convolution integral. Note that
$$\begin{aligned} y(t) &= \int_0^t G(t-\tau)u(\tau)\mathrm{d}\tau \\ &= \int_0^t G(\tau)u(t-\tau)\mathrm{d}\tau \quad \text{(Why is this step correct?)} \\ &= \int_0^t G(\tau)\delta(t-\tau)\mathrm{d}\tau \\ &= G(t), \ \ \forall t \geq 0. \end{aligned}$$

State-Space to Transfer Function

- Say we start with

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}_0 = \mathbf{x}(0), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t). \end{aligned}$$

- What's the transfer function $G(s)$ where $y(s) = G(s)u(s)$?
- Using LTs,

$$\begin{aligned} s\mathbf{x}(s) - \mathbf{x}_0 &= \mathbf{A}\mathbf{x}(s) + \mathbf{B}u(s), \\ s\mathbf{x}(s) - \mathbf{A}\mathbf{x}(s) &= \mathbf{B}u(s) + \mathbf{x}_0, \\ (s\mathbf{1} - \mathbf{A})\mathbf{x}(s) &= \mathbf{B}u(s) + \mathbf{x}_0, \\ \mathbf{x}(s) &= (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}u(s) + (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}_0. \end{aligned}$$

- Similarly, using LTs again,

$$\begin{aligned} y(s) = \mathbf{C}\mathbf{x}(s) + \mathbf{D}u(s) &= \mathbf{C}\left((s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}u(s) + (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}_0\right) + \mathbf{D}u(s) \\ &= \underbrace{\left(\mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)u(s)}_{\text{forced response}} + \underbrace{\mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}_0}_{\text{free response}}. \end{aligned}$$

- Thus, $G(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.

- Note that the time-domain solution of

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}_0 = \mathbf{x}(0),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t).\end{aligned}$$

  is

$$y(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\text{free response}} + \underbrace{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,\mathrm{d}\tau + \mathbf{D}u(t)}_{\text{forced response}},$$

  where $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}t)^k$ is the matrix exponential.

- As such,

$$\mathcal{L}\left\{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0\right\} = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}_0,$$

  and

$$\mathcal{L}\left\{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,\mathrm{d}\tau + \mathbf{D}u(t)\right\} = \left(\mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)u(s).$$

Python

In python,

- to go from state space to transfer function, use `control.ss2tf`, and
- to go from transfer function to state space, use `control.tf2ss`.

Normalization Revisited

- Consider

$$y(s) = G_1(s)u(s) + G_2(s)d(s), \qquad (1)$$

  which is assumed to be normalized.
  - $G_1(s) = G_{yu}(s)$ is the transfer function from the control input $u(s)$ to $y(s)$.
  - $G_2(s) = G_{yd}(s)$ is the transfer function from the disturbance $d(s)$ to $y(s)$.
  - Although $G_1(s) = G_2(s)$ in some situations (e.g., Figure 1), it's possible that $G_1(s) \neq G_2(s)$ (e.g., Figure 2).
- Consider

$$\tilde{y}(s) = \tilde{G}_1(s)\tilde{u}(s) + \tilde{G}_2(s)\tilde{d}(s). \qquad (2)$$

  which is not yet normalized.
- How does one go from (2), unnormalized, to (1), normalized?

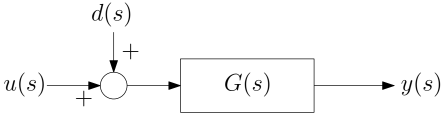

Figure 1: System with disturbance entering in the same way as the control input.

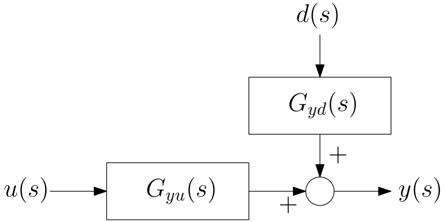

Figure 2: System with disturbance entering in the same way as the control input.

- Let u_{nor}, y_{nor}, d_{nor} be normalization constants such that

$$\tilde{u}(s) = u_{\text{nor}}\, u(s), \qquad \tilde{y}(s) = y_{\text{nor}}\, y(s), \qquad \tilde{d}(s) = d_{\text{nor}}\, d(s),$$

  where $\tilde{u}(s)$, $\tilde{y}(s)$, and $\tilde{d}(s)$ have units and $u(s)$, $y(s)$, $d(s)$ are unitless (normalized).
  - Almost always $u_{\text{nor}} = u_{\max}$, $y_{\text{nor}} = y_{\max}$, and $d_{\text{nor}} = d_{\max}$.
- It follows that

$$\begin{aligned}
\tilde{y}(s) &= \tilde{G}_1(s)\tilde{u}(s) + \tilde{G}_2(s)\tilde{d}(s), \\
y_{\text{nor}}\, y(s) &= \tilde{G}_1(s) u_{\text{nor}}\, u(s) + \tilde{G}_2(s) d_{\text{nor}}\, d(s), \\
y(s) &= \underbrace{y_{\text{nor}}^{-1}\, \tilde{G}_1(s) u_{\text{nor}}}_{G_1(s)}\, u(s) + \underbrace{y_{\text{nor}}^{-1}\, \tilde{G}_2(s) d_{\text{nor}}}_{G_2(s)}\, d(s).
\end{aligned}$$

Notes and References

Notes
See Appendix A and Sections 2.3 of [1], as well as [2] Chapter 2, and Section 1.4 of [3].

Additionally, these notes are partially based on / inspired by notes by Prof. M. Peet, ASU, Prof. C.J. Damaren, UTIAS, Prof. T.D. Barfoot, UTAIS, and Prof. J. Hoagg, U. Kentucky.

[1] L. Qiu and K. Zhou, *Introduction to Feedback Control*. Upper Saddle River, NJ: Prentice-Hall, Inc., 2010.
[2] W. J. Palm, *System Dynamics*, 3rd. Toronto, ON: McGraw Hill, 2013.
[3] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control - Analysis and Design*, Second. Hoboken, NJ: John Wiley & Sons, Inc., 2005.