The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norms

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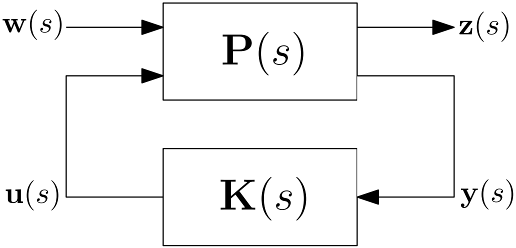
The Standard Control Problem

- Consider a **generalized plant**, $\mathbf{P}(s)$, connected in feedback with a **controller**, $\mathbf{K}(s)$.



- Inputs and outputs are as follows.
  - $\mathbf{w} \in \mathbb{R}^{n_w}$: exogenous inputs.
  - $\mathbf{z} \in \mathbb{R}^{n_z}$: regulated output, also called performance output.
  - $\mathbf{u} \in \mathbb{R}^{n_u}$: control input (controller output [applied by actuators]).
  - $\mathbf{y} \in \mathbb{R}^{n_y}$: plant output (controller input [a function of measurements]).
- Design $\mathbf{K}(s)$ to minimize (in terms of some sort of “cost/objective function”) the impact of $\mathbf{w}$ on $\mathbf{z}$.

The Control Design Problem

A very general, useful, and clever way to formulate a control design problem is to

$$\begin{array}{ll} \underset{\mathbf{K}(s)}{\text{minimize}} & \|\boldsymbol{\mathcal{T}}_{zw}\| \\ \text{such that} & \text{closed-loop system is asymptotically stable,} \end{array}$$

where $\mathbf{z}(t) = (\boldsymbol{\mathcal{T}}_{zw}\mathbf{w})(t)$ and $\mathbf{z}(s) = \mathbf{T}_{zw}(s)\mathbf{w}(s)$.

Given $\mathbf{y}(t) = (\boldsymbol{\mathcal{G}}\mathbf{u})(t)$, $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$, the most popular norms are:

- the $\mathcal{H}_2$ norm,

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right)\,\mathrm{d}t},$$

  where and $\mathbf{G}(t)$ is the impulse response of $\mathbf{G}(s)$; and

- the $\mathcal{H}_\infty$ norm,

$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\mathbf{G}(\jmath\omega)),$$

  where $\bar{\sigma}(\mathbf{G}(\jmath\omega)) = \sqrt{\bar{\lambda}(\mathbf{G}^\mathsf{H}(\jmath\omega)\mathbf{G}(\jmath\omega))}$.

Assume interconnection is well posed.

Okay, so why are these *system* norms so popular? How are they interpreted? How do we even compute them!?

System Responses

- Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(t_0) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{2}$$

where the *total response* in the time-domain is

$$\mathbf{y}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{x}_0 + \mathbf{C}\int_0^t \exp\big(\mathbf{A}(t-\tau)\big)\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t), \quad \forall t \geq 0. \tag{3}$$

- Equation (3) is often written as simply $\mathbf{y}(t) = (\mathcal{G}\mathbf{u})(t)$ where $\mathcal{G} : \mathcal{U} \to \mathcal{Y}$ is an *operator*.
- If $\mathbf{u}(t) = \mathbf{0}$ for all $t \geq 0$ then

$$\mathbf{y}_{\mathrm{free}}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{x}_0, \quad \forall t \geq 0,$$

which is the *free response* (a.k.a., the zero input response).

- If $\mathbf{x}_0 = \mathbf{0}$ then

$$\mathbf{y}_{\mathrm{forced}}(t) = \mathbf{C}\int_0^t \exp\big(\mathbf{A}(t-\tau)\big)\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t), \quad \forall t \geq 0,$$

which is the *forced response*.

Impulse Response

- Let $a < b$ where $a, b \in \mathbb{R}$. The impulse function is defined as

$$\int_a^b \delta(\tau)\,\mathrm{d}\tau = \begin{cases} 1, & a \leq 0 < b \\ 0, & \text{otherwise} \end{cases}$$

  or, more generally as

$$\int_a^b \delta(\tau - t_0)\mathbf{g}(\tau)\,\mathrm{d}\tau = \begin{cases} \mathbf{g}(t_0), & a \leq t_0 < b \\ 0, & \text{otherwise} \end{cases}$$

  where $t_0 \in [a, b]$ and $\mathbf{g} : [a, b] \to \mathbb{R}^n$ is continuous at t_0.
- Setting $\mathbf{u}(t) = \delta(t)\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^{n_u}$ is arbitrary, yields

$$\mathbf{y}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{x}_0 + \mathbf{G}(t)\mathbf{v}, \quad \forall t \geq 0,$$

  where

$$\mathbf{G}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B} + \mathbf{D}\delta(t), \quad \forall t \geq 0,$$

  is the *impulse response function*.
- When $\mathbf{u}(t) = \delta(t)\mathbf{v}$ the forced response is given by

$$\mathbf{y}(t) = \mathbf{G}(t)\mathbf{v} = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}\mathbf{v} + \delta(t)\mathbf{D}\mathbf{v}, \quad \forall t \geq 0,$$

  which is the *impulse response*.

Free versus Impulse Response

Fact

Let $\mathbf{D} = \mathbf{0}$, $n_u = 1$, and $\mathbf{x}_0 = \mathbf{B}v$. Then the free and impulse responses are equal to each other and are given by

$$\mathbf{y}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}v = \mathbf{C}\exp(\mathbf{A}t)\mathbf{x}_0, \quad \forall t \geq 0.$$

Proof.

The proof follows from a direct comparison of the free response and impulse response when $\mathbf{D} = \mathbf{0}$, $n_u = 1$, and $\mathbf{x}_0 = \mathbf{B}v$ where $\mathbf{B} \in \mathbb{R}^{n \times 1}$. $\square$

The impulse response can be interpreted as a free response with very specific initial conditions.

Signal Norms

- The L_2 norm of $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as

$$\|\mathbf{u}\|_2 = \sqrt{\int_0^\infty \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)\,\mathrm{d}t}.$$

- Using Parseval's Theorem,

$$\|\mathbf{u}\|_2 = \sqrt{\int_0^\infty \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)\,\mathrm{d}t} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{u}^\mathsf{H}(j\omega)\mathbf{u}(j\omega)\,\mathrm{d}\omega}$$

  where $\mathbf{u}(j\omega)$ is the Fourier transform of $\mathbf{u}(t)$.

- The truncation of $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^n$ is

$$\mathbf{u}_T = \begin{cases} \mathbf{u}(t), & 0 \le t \le T, \\ \mathbf{0}, & t > T \end{cases} \quad \forall T \in \mathbb{R}^+.$$

- The truncated L_2 norm is the L_2 norm of a truncated function,

$$\|\mathbf{u}\|_{2T} = \sqrt{\int_0^\infty \mathbf{u}_T^\mathsf{T}(t)\mathbf{u}_T(t)\,\mathrm{d}t} = \sqrt{\int_0^T \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)\,\mathrm{d}t}.$$

- The L_∞ norm of $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as $\|\mathbf{u}\|_\infty = \sup_{t\in\mathbb{R}^+}\left[\max_{i=1,\ldots,n}|u_i(t)|\right]$.

The $\mathcal{H}_2$ Norm

- Consider $\mathbf{y}(t) = (\boldsymbol{\mathcal{G}}\mathbf{u})(t)$ where $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$,

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B},$$

  $\mathbf{A}$ is Hurwitz, and $\mathbf{D} = \mathbf{0}$.

- The $\mathcal{H}_2$ norm is

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{G}^{\mathsf{T}}(t)\mathbf{G}(t)\right)\,\mathrm{d}t} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^\infty \operatorname{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right)\,\mathrm{d}\omega},$$

  where and $\mathbf{G}(t)$ is the impulse response of $\mathbf{G}(s)$.

- The matrix $\mathbf{A}$ *must be Hurwitz* for the $\mathcal{H}_2$ norm to be well defined.
- The matrix $\mathbf{D}$ *must equal zero* for the $\mathcal{H}_2$ norm to be well defined.

The $\mathcal{H}_2$ Norm Interpretations

- Time domain interpretations.
  - Impulse response interpretation: the square root of the sum of the L_2 norm squared of the forced response when an impulse is applied to each input channel individually.
  - Free response interpretation: the L_2 norm of the free response.
- Frequency domain interpretations.
  - SISO frequency domain interpretation: square root of the area under the magnitude (of the transfer function) squared.
  - MIMO frequency domain interpretation: square root of the sum of the area's under the singular values (of the transfer matrix) squared.
- Random signal interpretation.
  - In both SISO and MIMO cases, the RMS-value of the system output when the input is white noise.

Impulse Response Interpretation

- Let $\mathbf{1}_\imath \in \mathbb{R}^{n_u}$ define a column matrix with a 1 in the $\imath^{\text{th}}$ row, that is, $\mathbf{1}_\imath = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0 \;\; 0]^\mathsf{T}$.
- The input $\mathbf{u}(t) = \delta(t)\mathbf{1}_\imath$ yields the impulse response

$$\mathbf{y}_\imath(t) = \mathbf{G}(t)\mathbf{1}_\imath = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}\mathbf{1}_\imath$$

  when $\mathbf{D} = \mathbf{0}$.
- Consider

$$\begin{aligned}\sum_{\imath=1}^{n_u} \|\mathbf{y}_\imath\|_2^2 &= \sum_{\imath=1}^{n_u} \int_0^\infty \mathbf{y}_\imath^\mathsf{T}(t)\mathbf{y}_\imath(t)\,\mathrm{d}t \\ &= \int_0^\infty \sum_{\imath=1}^{n_u} \left(\mathbf{G}(t)\mathbf{1}_\imath\right)^\mathsf{T}\left(\mathbf{G}(t)\mathbf{1}_\imath\right)\,\mathrm{d}t \\ &= \int_0^\infty \sum_{\imath=1}^{n_u} \mathbf{1}_\imath^\mathsf{T}\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\mathbf{1}_\imath\,\mathrm{d}t,\end{aligned}$$

  which is the sum of the L_2 norm squared of the $\imath = 1, 2, \ldots, n_u$ impulse responses.
- Note that for $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathrm{tr}(\mathbf{A})$ can be written

$$\mathrm{tr}(\mathbf{A}) = \sum_{\imath=1}^{n} a_{\imath\imath} = \sum_{\imath=1}^{n} \mathbf{1}_\imath^\mathsf{T}\mathbf{A}\mathbf{1}_\imath.$$

- Therefore,

$$\sum_{\imath=1}^{n_u} \|\mathbf{y}_\imath\|_2^2 = \int_0^\infty \sum_{\imath=1}^{n_u} \mathbf{1}_\imath^\mathsf{T} \mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\mathbf{1}_\imath \, \mathrm{d}t$$
$$= \int_0^\infty \mathrm{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right) \mathrm{d}t,$$

and by taking the square root of both sides gives

$$\sqrt{\sum_{\imath=1}^{n_u} \|\mathbf{y}_\imath\|_2^2} = \sqrt{\int_0^\infty \mathrm{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right) \mathrm{d}t} = \|\boldsymbol{\mathcal{G}}\|_2 \, .$$

- It follows that the $\mathcal{H}_2$ norm is equal to the square root of the sum of the L_2 norm squared of the forced response when an impulse is applied to each input channel individually.

Free Response Interpretation

- The free response with initial condition $\mathbf{x}_0 = \mathbf{B} \in \mathbb{R}^{n_u \times 1}$ is

$$\mathbf{y}(t) = \mathbf{G}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}.$$

- The L_2 norm of the free response with initial condition $\mathbf{x}_0 = \mathbf{B} \in \mathbb{R}^{n_u \times 1}$ is

$$\begin{aligned}
\|\mathbf{y}\|_2 &= \sqrt{\int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)\,\mathrm{d}t} \\
&= \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)\right)\,\mathrm{d}t} \\
&= \sqrt{\int_0^\infty \operatorname{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right)\,\mathrm{d}t} \\
&= \|\boldsymbol{\mathcal{G}}\|_2\,.
\end{aligned}$$

- Therefore, the $\mathcal{H}_2$ norm is equal to the L_2 norm of the free response with initial condition $\mathbf{x}_0 = \mathbf{B} \in \mathbb{R}^{n_u \times 1}$.

SISO Frequency Domain Interpretation

- Consider the SISO case, $y(s) = G(s)u(s)$, where
$$G(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}.$$
- When $n_y = n_u = 1$ the term
$$\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right)\, \mathrm{d}\omega}$$
becomes
$$\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} G^{\mathsf{H}}(j\omega)G(j\omega)\, \mathrm{d}\omega.}$$
- Also, when $n_y = n_u = 1$ the term $G^{\mathsf{H}}(j\omega)G(j\omega)$ can be written
$$\begin{aligned} G^{\mathsf{H}}(j\omega)G(j\omega) &= \big(\mathrm{Re}\{G(j\omega)\} - j\mathrm{Im}\{G(j\omega)\}\big)\big(\mathrm{Re}\{G(j\omega)\} + j\mathrm{Im}\{G(j\omega)\}\big) \\ &= (\mathrm{Re}\{G(j\omega)\})^2 + (\mathrm{Im}\{G(j\omega)\})^2 \\ &= |G(j\omega)|^2, \end{aligned}$$
which is the transfer function magnitude squared.

- The $\mathcal{H}_2$ norm is equal to

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G^{\mathsf{H}}(j\omega)G(j\omega)\right)\,\mathrm{d}\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\,\mathrm{d}\omega}, \tag{4}$$

which can be interpreted as the square root of the area under $|G(j\omega)|^2$ over all frequencies.

MIMO Frequency Domain Interpretation

- Recall that any matrix can be written using a singular value decomposition (SVD).
- In particular, the (not necessarily square nor real) matrix $\mathbf{G}$ can be written

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}},$$

  where $\mathbf{U}$ and $\mathbf{V}$ are unitary (i.e., $\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathsf{H}} = \mathbf{1}$) and $\boldsymbol{\Sigma} = \text{diag}\left\{\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}\right\}$.
- By writing the transfer matrix $\mathbf{G}(j\omega)$ as

$$\mathbf{G}(j\omega) = \mathbf{U}(j\omega)\boldsymbol{\Sigma}(j\omega)\mathbf{V}^{\mathsf{H}}(j\omega),$$

  the expression $\operatorname{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right)$ can be written

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right) &= \operatorname{tr}\left(\mathbf{V}(j\omega)\boldsymbol{\Sigma}^{\mathsf{H}}(j\omega)\mathbf{U}^{\mathsf{H}}(j\omega)\mathbf{U}(j\omega)\boldsymbol{\Sigma}(j\omega)\mathbf{V}^{\mathsf{H}}(j\omega)\right) \\
&= \operatorname{tr}\left(\boldsymbol{\Sigma}^{\mathsf{H}}(j\omega)\boldsymbol{\Sigma}(j\omega)\mathbf{V}^{\mathsf{H}}(j\omega)\mathbf{V}(j\omega)\right) \\
&= \operatorname{tr}\left(\boldsymbol{\Sigma}^{\mathsf{H}}(j\omega)\boldsymbol{\Sigma}(j\omega)\right) \\
&= \sum_{i=1}^{\min(n_y,n_u)} \sigma_i^2(\mathbf{G}(j\omega)),
\end{aligned}$$

  where the “wrap around” property of trace has been employed.

- It then follows that

$$
\begin{aligned}
\|\boldsymbol{\mathcal{G}}\|_2 &= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right)\, \mathrm{d}\omega} \\
&= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{i=1}^{\min(n_y,n_u)} \sigma_i^2(\mathbf{G}(j\omega))\,\mathrm{d}\omega} \\
&= \sqrt{\sum_{i=1}^{\min(n_y,n_u)} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_i^2(\mathbf{G}(j\omega))\,\mathrm{d}\omega\right]}.
\end{aligned} \tag{5}
$$

- The $\mathcal{H}_2$ is equal the square root of the of the sum of the area under each singular value (of the transfer matrix $\mathbf{G}(s)$) squared.

Random Signals

- Some functions, such as $\mathbf{u}(t)$ where $u_i(t) = \bar{u}_i \sin(\omega_i t + \phi_i)$, do not have a finite L_2 norm.
- To measure the size of such functions, the *root-mean-square* (RMS) value fo the function is used, where

$$\|\mathbf{u}\|_{\mathrm{rms}} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t)\,\mathrm{d}t}.$$

- $\|\mathbf{u}\|_{\mathrm{rms}}^2$ represents the average power of the signal.
- $\|\mathbf{u}\|_{\mathrm{rms}}^2$ is also equal to

$$\|\mathbf{u}\|_{\mathrm{rms}} = \sqrt{\mathrm{tr}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{S}_u(\omega)\,\mathrm{d}\omega\right)} = \sqrt{\mathrm{tr}\left(\mathbf{R}_u(0)\right)}.$$

  - The power spectral density of $\mathbf{u}(t)$ is $\mathbf{S}_u(\omega) = \int_{-\infty}^{\infty} \mathbf{R}_u(\tau) e^{-\jmath\omega\tau}\,\mathrm{d}\tau$.
  - The auto-correlation matrix of $\mathbf{u}(t)$ is $\mathbf{R}_u(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{u}(t)\mathbf{u}^{\mathsf{T}}(t+\tau)\,\mathrm{d}\tau$. $\mathbf{R}_u(0)$ is often referred to as the *covariance matrix* of $\mathbf{u}(t)$.

SISO PSD Interpretation

- Consider the SISO case, $y(s) = G(s)u(s)$.
- It can be shown that

$$S_y(\omega) = |G(j\omega)|^2 S_u(\omega),$$

  meaning the PSD of the input $S_u(\omega)$ is amplified or attenuated by $|G(j\omega)|^2$ to give the PSD of the output $S_y(\omega)$.
- It is customary to assume $S_u(\omega) = 1$, $\forall \omega \in \mathbb{R}$, meaning the input $u(t)$ is a *white noise* signal. In this case,

$$S_y(\omega) = |G(j\omega)|^2,$$

  meaning the output PSD $S_y(\omega)$ is equal to $|G(j\omega)|^2$ at each frequency.
- Further, notice that

$$\|y\|_{\mathrm{rms}} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\,\mathrm{d}\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\,\mathrm{d}\omega},$$

  which corresponds exactly to (4).
- As such, can interpret the $\mathcal{H}_2$ norm as the RMS-value of the system output when the input is white noise.

MIMO PSD Interpretation

- Consider the MIMO case, $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$.
- It can be shown that

$$\mathbf{S}_y(\omega) = \mathbf{G}(\jmath\omega)\mathbf{S}_u(\omega)\mathbf{G}^{\mathsf{H}}(\jmath\omega),$$

  where the PSD of the input is $\mathbf{S}_u(\omega)$ and the output PSD is $\mathbf{S}_y(\omega)$.
- It is customary to assume $\mathbf{S}_u(\omega) = \mathbf{1}$, $\forall \omega \in \mathbb{R}$, meaning the input $\mathbf{u}(t)$ is a *white noise* signal. In this case,

$$\mathbf{S}_y(\omega) = \mathbf{G}(\jmath\omega)\mathbf{G}^{\mathsf{H}}(\jmath\omega).$$

- Further, notice that

$$\|\mathbf{y}\|_{\mathrm{rms}} = \sqrt{\mathrm{tr}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{S}_y(\omega)\,\mathrm{d}\omega\right)} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}\left(\mathbf{G}(j\omega)\mathbf{G}^{\mathsf{H}}(j\omega)\right)\,\mathrm{d}\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}\left(\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right)\,\mathrm{d}\omega},$$

  which corresponds exactly to (5).
- Again, can interpret the $\mathcal{H}_2$ norm as the RMS-value of the system output when the input is white noise.

The $\mathcal{H}_\infty$ Norm

- Consider $\mathbf{y}(t) = (\boldsymbol{\mathcal{G}}\mathbf{u})(t)$ where $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$,

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D},$$

  and $\mathbf{A}$ is Hurwitz.
- The $\mathcal{H}_\infty$ norm is

$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\mathbf{u}\in L_2, \mathbf{u}\neq\mathbf{0}} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\mathbf{G}(\jmath\omega)),$$

  where $\bar{\sigma}(\mathbf{G}(\jmath\omega)) = \sqrt{\bar{\lambda}(\mathbf{G}^{\mathsf{H}}(\jmath\omega)\mathbf{G}(\jmath\omega))}$.
- The matrix $\mathbf{A}$ *must be Hurwitz* for the $\mathcal{H}_\infty$ norm to be well defined.
- The matrix $\mathbf{D}$ does not have to equal zero for the $\mathcal{H}_\infty$ norm to be well defined (although, it may happen to equal zero).

The $\mathcal{H}_\infty$ Norm Interpretations

- Time domain interpretations.
  - The "worst case" ratio of $\|\mathbf{y}\|_2$ to $\|\mathbf{u}\|_2$.
- Frequency domain interpretations.
  - SISO frequency domain interpretation: supremum (i.e., peak) on magnitude part of Bode diagram.
  - MIMO frequency domain interpretation: supremum of the maximum singular value associated with transfer matrix $\mathbf{G}(s)$.

The "Worst Case" Ratio of $\|\mathbf{y}\|_2$ to $\|\mathbf{u}\|_2$

- The squared L_2 norm of $\mathbf{y}$ is

$$\|\mathbf{y}\|_2^2 = \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)dt = \tfrac{1}{2\pi}\int_{-\infty}^\infty \mathbf{y}^\mathsf{H}(j\omega)\mathbf{y}(j\omega)\,\mathrm{d}\omega,$$

  where Parseval's Theorem has been used to transfer into the frequency domain.

- Using the relation $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$ it follows that

$$\begin{aligned}
\|\mathbf{y}\|_2^2 &= \tfrac{1}{2\pi}\int_{-\infty}^\infty \mathbf{u}^\mathsf{H}(j\omega)\mathbf{G}^\mathsf{H}(j\omega)\mathbf{G}(j\omega)\mathbf{u}(j\omega)\,\mathrm{d}\omega \\
&\le \underbrace{\sup_{\omega\in\mathbb{R}} \bar{\sigma}^2(\mathbf{G}(j\omega))}_{\|\boldsymbol{\mathcal{G}}\|_\infty^2} \tfrac{1}{2\pi}\int_{-\infty}^\infty \mathbf{u}^\mathsf{H}(j\omega)\mathbf{u}(j\omega)\,\mathrm{d}\omega \\
&= \|\boldsymbol{\mathcal{G}}\|_\infty^2 \, \|\mathbf{u}\|_2^2\,.
\end{aligned}$$

- Taking the square root of both sides yields

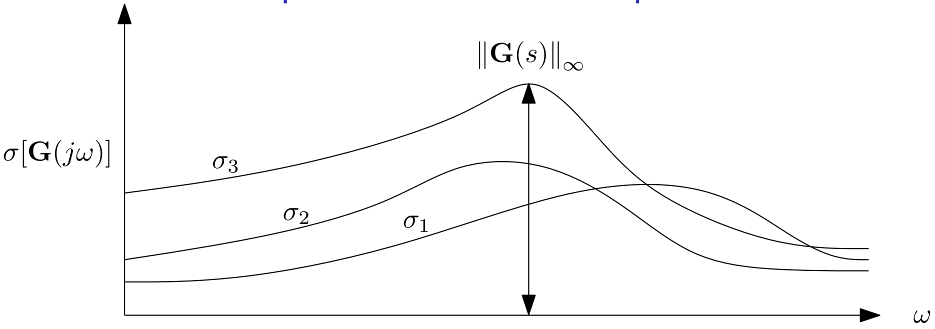
$$\|\mathbf{y}\|_2 \leq \|\boldsymbol{\mathcal{G}}\|_\infty \|\mathbf{u}\|_2\,, \qquad \text{or} \qquad \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} \leq \|\boldsymbol{\mathcal{G}}\|_\infty\,, \quad \|\mathbf{u}\|_2 \neq 0.$$

- The "worst case" ratio of $\|\mathbf{y}\|_2$ to $\|\mathbf{u}\|_2$ leads to

$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\mathbf{u}\in L_2, \|\mathbf{u}\|_2 \neq 0} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} = \sup_{\mathbf{u}\in L_2, \|\mathbf{u}\|_2 \neq 0} \frac{\|\boldsymbol{\mathcal{G}}\mathbf{u}\|_2}{\|\mathbf{u}\|_2} = \sup_{\mathbf{u}\in L_2, \|\mathbf{u}\|_2 = 1} \|\boldsymbol{\mathcal{G}}\mathbf{u}\|_2\,,$$

meaning the $\mathcal{H}_\infty$ norm is the induced norm on L_2.

SISO and MIMO Interpretations in the Freq. Domain



- Recall that
$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\mathbf{G}(j\omega)).$$
- In the SISO case, $\bar{\sigma}(\mathbf{G}(j\omega))$ is equivalent to $|G(j\omega)|$.
- Therefore, in the SISO case,
$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\omega\in\mathbb{R}} |G(j\omega)|.$$

How to actually **compute** the $\mathcal{H}_2$ norm?

To compute the $\mathcal{H}_2$ norm, we need the following theorem.

Theorem

Assume $\mathbf{A}$ *is Hurwitz, and consider the Lyapunov equation*

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{Q} = \mathbf{0},$$

where $\mathbf{Q} = \mathbf{Q}^\mathsf{T}$*. The following statements hold.*

1. $\mathbf{P} = \int_0^\infty \exp(\mathbf{A}^\mathsf{T} t)\mathbf{Q}\exp(\mathbf{A}t)\,\mathrm{d}t$.
2. $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *if* $\mathbf{Q} = \mathbf{Q}^\mathsf{T} > 0$; $\mathbf{P} \geq 0$ *if* $\mathbf{Q} = \mathbf{Q}^\mathsf{T} \geq 0$.
3. *Let* $\mathbf{Q} = \mathbf{Q}^\mathsf{T} \geq 0$*. Then,* $(\mathbf{A}, \mathbf{Q})$ *is observable if and only if* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$.

- As a consequence of part 3 of the above theorem, given $\mathbf{A}$ that is Hurwitz, $(\mathbf{A}, \mathbf{C})$ is observable if and only if $\mathbf{M} = \mathbf{M}^\mathsf{T} > 0$ is the solution to

  $$\mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0},$$

  where $\mathbf{M}$ is the (infinite horizon) observability Gramian. Note that

  $$\mathbf{M} = \int_0^\infty \exp(\mathbf{A}^\mathsf{T} t)\mathbf{C}^\mathsf{T}\mathbf{C}\exp(\mathbf{A}t)\,\mathrm{d}t.$$

- Similarly, from part 3 again, given $\mathbf{A}$ that is Hurwitz, $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $\mathbf{W} = \mathbf{W}^\mathsf{T} > 0$ is the solution to

  $$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{0},$$

  where $\mathbf{W}$ is the (infinite horizon) controllability Gramian. Note that

  $$\mathbf{W} = \int_0^\infty \exp(\mathbf{A}t)\mathbf{B}\mathbf{B}^\mathsf{T}\exp(\mathbf{A}^\mathsf{T} t)\,\mathrm{d}t.$$

- Recall that the $\mathcal{H}_2$ norm is

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\int_0^\infty \mathrm{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right)\,\mathrm{d}t},$$

  where $\mathbf{G}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}$ is the impulse response of $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$, $\mathbf{A}$ is Hurwitz, and $\mathbf{D} = \mathbf{0}$. Assume $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{C})$ is observable.
- It follows that

$$\begin{aligned}
\|\boldsymbol{\mathcal{G}}\|_2^2 &= \int_0^\infty \mathrm{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{tr}\left(\mathbf{B}^\mathsf{T}\exp(\mathbf{A}^\mathsf{T}t)\mathbf{C}^\mathsf{T}\mathbf{C}\exp(\mathbf{A}t)\mathbf{B}\right)\,\mathrm{d}t \\
&= \mathrm{tr}\left[\mathbf{B}^\mathsf{T}\left(\int_0^\infty \exp(\mathbf{A}^\mathsf{T}t)\mathbf{C}^\mathsf{T}\mathbf{C}\exp(\mathbf{A}t)\,\mathrm{d}t\right)\mathbf{B}\right] \\
&= \mathrm{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{M}\mathbf{B}\right),
\end{aligned}$$

  where $\mathbf{M} = \mathbf{M}^\mathsf{T} > 0$ is the solution to the Lyapunov equation $\mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0}$.
- Therefore, the $\mathcal{H}_2$ norm is

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\mathrm{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{M}\mathbf{B}\right)},$$
$$\mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0}.$$

- There is yet another way to compute $\mathcal{H}_2$ norm. Using the "wrap around" property of trace, write $\|\mathcal{G}\|_2$ as

$$\|\mathcal{G}\|_2 = \sqrt{\int_0^\infty \mathrm{tr}\left(\mathbf{G}^\mathsf{T}(t)\mathbf{G}(t)\right)\,\mathrm{d}t} = \sqrt{\int_0^\infty \mathrm{tr}\left(\mathbf{G}(t)\mathbf{G}^\mathsf{T}(t)\right)\,\mathrm{d}t},$$

  where, as before, $\mathbf{G}(t) = \mathbf{C}\exp(\mathbf{A}t)\mathbf{B}$ is the impulse response of $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$, $\mathbf{A}$ is Hurwitz, and $\mathbf{D} = \mathbf{0}$. Assume $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{C})$ is observable.
- It follows that

$$\begin{aligned}
\|\mathcal{G}\|_2^2 &= \int_0^\infty \mathrm{tr}\left(\mathbf{G}(t)\mathbf{G}^\mathsf{T}(t)\right)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{tr}\left(\mathbf{C}\exp(\mathbf{A}t)\mathbf{B}\mathbf{B}^\mathsf{T}\exp(\mathbf{A}^\mathsf{T}t)\mathbf{C}^\mathsf{T}\right)\,\mathrm{d}t \\
&= \mathrm{tr}\left[\mathbf{C}\left(\int_0^\infty \exp(\mathbf{A}t)\mathbf{B}\mathbf{B}^\mathsf{T}\exp(\mathbf{A}^\mathsf{T}t)\,\mathrm{d}t\right)\mathbf{C}^\mathsf{T}\right] \\
&= \mathrm{tr}\left(\mathbf{C}\mathbf{W}\mathbf{C}^\mathsf{T}\right),
\end{aligned}$$

  where $\mathbf{W} = \mathbf{W}^\mathsf{T} > 0$ is the solution to the Lyapunov equation $\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{0}$.
- Therefore, the $\mathcal{H}_2$ norm is

$$\|\mathcal{G}\|_2 = \sqrt{\mathrm{tr}\left(\mathbf{C}\mathbf{W}\mathbf{C}^\mathsf{T}\right)},$$
$$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{0}.$$

Using Convex Optimization and LMIs to Compute the $\mathcal{H}_2$ Norm

Theorem

Consider $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$ *where* $\mathbf{A}$ *is Hurwitz,* $(\mathbf{A}, \mathbf{B})$ *is controllable, and* $(\mathbf{A}, \mathbf{C})$ *is observable. The following conditions are equivalent.*

1. $\|\mathcal{G}\|_2 < \nu$.
2. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\begin{aligned} \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) &< \nu^2, \\ \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} &< 0. \end{aligned}$$

3. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\begin{aligned} \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}\right) &< \nu^2, \\ \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} &< 0. \end{aligned}$$

As always, identify the p and the q (and, in this case, the r).

- $p = \left(\|\boldsymbol{\mathcal{G}}\|_2 < \nu \right)$.
- $q = \big($ There exists $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ such that $\operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) < \nu^2$, $\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} < 0\big)$.
- $r = \big($ There exists $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ such that $\operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}\right) < \nu^2$, $\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} < 0\big)$.

$q \Rightarrow p$

- q being true means that $\mathbf{P} = \mathbf{P}^\top > 0$ such that $\mathbf{PA} + \mathbf{A}^\top \mathbf{P} + \mathbf{C}^\top \mathbf{C} < 0$.
- Define $-\mathbf{Q} = \mathbf{PA} + \mathbf{A}^\top \mathbf{P} + \mathbf{C}^\top \mathbf{C}$ where $\mathbf{Q} = \mathbf{Q}^\top > 0$.
- Owing to the fact that $\mathbf{A}$ is Hurwitz, there exists a unique solution $\mathbf{E} = \mathbf{E}^\top > 0$ to the Lyapunov equation

$$\mathbf{EA} + \mathbf{A}^\top \mathbf{E} = -\mathbf{Q}.$$

- Notice that

$$\begin{aligned}\mathbf{0} &= -\mathbf{Q} - (-\mathbf{Q}) \\ &= \mathbf{PA} + \mathbf{A}^\top \mathbf{P} + \mathbf{C}^\top \mathbf{C} - (\mathbf{EA} + \mathbf{A}^\top \mathbf{E}) \\ &= (\mathbf{P} - \mathbf{E})\mathbf{A} + \mathbf{A}^\top (\mathbf{P} - \mathbf{E}) + \mathbf{C}^\top \mathbf{C}.\end{aligned}$$

- Because $(\mathbf{A}, \mathbf{C})$ is observable it follows that the matrix $\mathbf{M} = \mathbf{P} - \mathbf{E} > 0$ is the solution to $\mathbf{MA} + \mathbf{A}^\top \mathbf{M} + \mathbf{C}^\top \mathbf{C} = \mathbf{0}$.
- Therefore, by using $\operatorname{tr}\left(\mathbf{B}^\top \mathbf{PB}\right) < \nu^2$ it follows that

$$\operatorname{tr}\left(\mathbf{B}^\top \mathbf{MB}\right) = \operatorname{tr}\left(\mathbf{B}^\top \mathbf{PB}\right) - \operatorname{tr}\left(\mathbf{B}^\top \mathbf{EB}\right) < \nu^2 - \underbrace{\operatorname{tr}\left(\mathbf{B}^\top \mathbf{EB}\right)}_{\geq 0} < \nu^2.$$

- Thus, q implies $\operatorname{tr}\left(\mathbf{B}^\top \mathbf{MB}\right) < \nu^2$ where $\mathbf{M} = \mathbf{M}^\top > 0$ is the solution of $\mathbf{MA} + \mathbf{A}^\top \mathbf{M} + \mathbf{C}^\top \mathbf{C} = \mathbf{0}$, which is equivalent to $\|\boldsymbol{\mathcal{G}}\|_2 < \nu$, which is p.

$p \Rightarrow q$

- p being true means $\|\mathcal{G}\|_2 < \nu$, which means that $\operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{M}\mathbf{B}\right) < \nu^2$ where $\mathbf{M} = \mathbf{M}^\mathsf{T} > 0$ is the solution of $\mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0}$.
- Owing to the fact that $\mathbf{A}$ is Hurwitz, there exists $\mathbf{M}_0 = \mathbf{M}_0^\mathsf{T} > 0$ satisfying

$$\mathbf{M}_0\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M}_0 < 0.$$

- Let $\mathbf{P} = \mathbf{M} + \epsilon\mathbf{M}_0 > 0$ where $\epsilon > 0$ is a small number. Notice that

$$\begin{aligned} & \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{M}\mathbf{B}\right) < \nu^2, \\ \Leftrightarrow \quad & \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) - \epsilon\operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{M}_0\mathbf{B}\right) < \nu^2, \\ & \Rightarrow \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) < \nu^2 \quad \text{for sufficiently small } \epsilon > 0. \end{aligned}$$

- Also, notice that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} = \underbrace{\mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C}}_{=\mathbf{0}} + \epsilon\underbrace{\left(\mathbf{M}_0\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M}_0\right)}_{<0} < 0.$$

- Thus, p implies there exists $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ such that $\operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) < \nu^2$, $\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} < 0$, which is q.

- To find the smallest ν such that $\|\mathcal{G}\|_2 < \nu$, solve the following convex optimization problem subject to LMI constraints:

$$\begin{aligned} \underset{\mathbf{P}=\mathbf{P}^\top}{\text{minimize}} \quad & \operatorname{tr}\left(\mathbf{B}^\top \mathbf{P} \mathbf{B}\right) \\ \text{such that} \quad & \mathbf{P} = \mathbf{P}^\top > 0, \\ & \mathbf{P}\mathbf{A} + \mathbf{A}^\top \mathbf{P} + \mathbf{C}^\top \mathbf{C} < 0. \end{aligned}$$

  The smallest ν is then $\nu = \sqrt{\operatorname{tr}\left(\mathbf{B}^\top \mathbf{P} \mathbf{B}\right)}$.

- Alternatively, the smallest ν such that $\|\mathcal{G}\|_2 < \nu$ can be found by solve the following convex optimization problem subject to LMI constraints:

$$\begin{aligned} \underset{\mathbf{P}=\mathbf{P}^\top}{\text{minimize}} \quad & \operatorname{tr}\left(\mathbf{C} \mathbf{P} \mathbf{C}^\top\right) \\ \text{such that} \quad & \mathbf{P} = \mathbf{P}^\top > 0, \\ & \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top < 0. \end{aligned}$$

  The smallest ν is then $\nu = \sqrt{\operatorname{tr}\left(\mathbf{C} \mathbf{P} \mathbf{C}^\top\right)}$.

How to actually **compute** the $\mathcal{H}_\infty$ norm?

In this case, we'll jump right to the context optimization + LMI formulation.

Theorem

Consider $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ *where* $\mathbf{A}$ *is Hurwitz,* $(\mathbf{A}, \mathbf{B})$ *is controllable, and* $(\mathbf{A}, \mathbf{C})$ *is observable. The following conditions are equivalent.*

1. $\|\mathcal{G}\|_\infty < \gamma$.
2. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\mathbf{F} = \begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} + \mathbf{C}^\mathsf{T}\mathbf{D} \\ * & \mathbf{D}^\mathsf{T}\mathbf{D} - \gamma^2\mathbf{1} \end{bmatrix} < 0.$$

Note, the "$*$" is just the symmetric part of the matrix in question.

Also, this is a *feasibility problem*. The γ is not necessarily "the smallest" (i.e., "inf").

As always, identify the p and the q.

- $p = \big(\|\mathcal{G}\|_\infty < \gamma \big)$.
- $q = \big($ There exists $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ such that $\mathbf{F} < 0 \big)$.

$q \Rightarrow p$

- Only $q \Rightarrow p$ will be shown.
- The approach that will be taken to show $q \Rightarrow p$ is to show that

$$\sup_{\mathbf{u} \in L_2, \mathbf{u} \neq \mathbf{0}} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} < \gamma$$

  using a Lyapunov-like analysis, where

$$\|\mathcal{G}\|_\infty = \sup_{\mathbf{u} \in L_2, \mathbf{u} \neq \mathbf{0}} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2},$$

  which in turn implies that $\|\mathcal{G}\|_\infty < \gamma < \infty$.
- To start, consider the Lyapunov-like function

$$V(\mathbf{x}(t)) = \mathbf{x}^\mathsf{T}(t)\mathbf{P}\mathbf{x}(t),$$

  where $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$.

- Recall that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(t_0)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).$$

- The time-rate-of-change of V is

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \dot{\mathbf{x}}^{\mathsf{T}}(t)\mathbf{P}\mathbf{x}(t) + \mathbf{x}^{\mathsf{T}}(t)\mathbf{P}\dot{\mathbf{x}}(t) \\
&= \mathbf{x}^{\mathsf{T}}(t)\left(\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P}\right)\mathbf{x}(t) + \mathbf{x}^{\mathsf{T}}(t)\mathbf{P}\mathbf{B}\mathbf{u}(t) + \mathbf{u}^{\mathsf{T}}(t)\mathbf{B}^{\mathsf{T}}\mathbf{P}\mathbf{x}(t) \\
&\quad + \mathbf{y}^{\mathsf{T}}(t)\mathbf{y}(t) - \mathbf{y}^{\mathsf{T}}(t)\mathbf{y}(t) + \gamma^2\mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t) - \gamma^2\mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t) \\
&= \mathbf{x}^{\mathsf{T}}(t)\left(\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P} + \mathbf{C}^{\mathsf{T}}\mathbf{C}\right)\mathbf{x}(t) \\
&\quad + \mathbf{x}^{\mathsf{T}}(t)(\mathbf{P}\mathbf{B} + \mathbf{C}^{\mathsf{T}}\mathbf{D})\mathbf{u}(t) + \mathbf{u}^{\mathsf{T}}(t)(\mathbf{B}^{\mathsf{T}}\mathbf{P} + \mathbf{D}^{\mathsf{T}}\mathbf{C}^{\mathsf{T}})\mathbf{x}(t) \\
&\quad + \mathbf{u}^{\mathsf{T}}(t)(\mathbf{D}^{\mathsf{T}}\mathbf{D} - \gamma^2\mathbf{1})\mathbf{u}(t) - \mathbf{y}^{\mathsf{T}}(t)\mathbf{y}(t) + \gamma^2\mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t) \\
&= \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P} + \mathbf{C}^{\mathsf{T}}\mathbf{C} & \mathbf{P}\mathbf{B} + \mathbf{C}^{\mathsf{T}}\mathbf{D} \\ * & \mathbf{D}^{\mathsf{T}}\mathbf{D} - \gamma^2\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&\quad - \mathbf{y}^{\mathsf{T}}(t)\mathbf{y}(t) + \gamma^2\mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t).
\end{aligned}$$

- Because $\mathbf{F} < 0$ it follows that

$$\dot{V}(\mathbf{x}(t)) < -\mathbf{y}^{\mathsf{T}}(t)\mathbf{y}(t) + \gamma^2\mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t).$$

- Integrating both sides between $t = 0$ and $t = T$ and rearranging yields

$$\begin{aligned} V(\mathbf{x}(T)) - V(\mathbf{x}(0)) &< -\int_0^T \mathbf{y}^{\mathsf{T}}(\tau)\mathbf{y}(\tau)\,\mathrm{d}\tau + \gamma^2 \int_0^T \mathbf{u}^{\mathsf{T}}(\tau)\mathbf{u}(\tau)\,\mathrm{d}\tau, \\ V(\mathbf{x}(T)) + \int_0^T \mathbf{y}^{\mathsf{T}}(\tau)\mathbf{y}(\tau)\,\mathrm{d}\tau &< \gamma^2 \int_0^T \mathbf{u}^{\mathsf{T}}(\tau)\mathbf{u}(\tau)\,\mathrm{d}\tau + V(\mathbf{x}(0)) \\ \|\mathbf{y}\|_{2T}^2 &< \gamma^2 \|\mathbf{u}\|_{2T}^2 + V(\mathbf{x}(0)) \end{aligned}$$

- When working with a transfer function $\mathbf{G}(s)$ we assume ICs are quiescent; invoking this assumption leads to

$$\begin{aligned} \|\mathbf{y}\|_{2T}^2 &< \gamma^2 \|\mathbf{u}\|_{2T}^2 , \\ \|\mathbf{y}\|_{2T} &< \gamma \|\mathbf{u}\|_{2T} , \\ \frac{\|\mathbf{y}\|_{2T}}{\|\mathbf{u}\|_{2T}} &< \gamma. \end{aligned}$$

- Letting $T \to \infty$ and taking the “worst case” ratio of $\|\mathbf{y}\|_2$ to $\|\mathbf{u}\|_2$ leads to

$$\sup_{\mathbf{u} \in L_2, \mathbf{u} \neq \mathbf{0}} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} < \gamma.$$

Equivalent LMIs

Theorem (Schur Complement Lemma)

Let the symmetric matrix $\mathbf{A}$ be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^{\mathsf{T}} & \mathbf{A}_{22} \end{bmatrix}.$$

Then,

$$\begin{aligned} \mathbf{A} < 0 \quad &\Leftrightarrow \quad \mathbf{A}_{11} < 0, \quad \mathbf{A}_{22} - \mathbf{A}_{12}^{\mathsf{T}}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} < 0, \\ &\Leftrightarrow \quad \mathbf{A}_{22} < 0, \quad \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\mathsf{T}} < 0, \end{aligned}$$

and

$$\begin{aligned} \mathbf{A} > 0 \quad &\Leftrightarrow \quad \mathbf{A}_{11} > 0, \quad \mathbf{A}_{22} - \mathbf{A}_{12}^{\mathsf{T}}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} > 0, \\ &\Leftrightarrow \quad \mathbf{A}_{22} > 0, \quad \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\mathsf{T}} > 0. \end{aligned}$$

Theorem

The following are equivalent.

1. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\mathbf{F}_1 = \begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} + \mathbf{C}^\mathsf{T}\mathbf{D} \\ * & \mathbf{D}^\mathsf{T}\mathbf{D} - \gamma^2\mathbf{1} \end{bmatrix} < 0.$$

2. *There exists* $\bar{\mathbf{P}} = \bar{\mathbf{P}}^\mathsf{T} > 0$ *such that*

$$\mathbf{F}_2 = \begin{bmatrix} \bar{\mathbf{P}}\mathbf{A} + \mathbf{A}^\mathsf{T}\bar{\mathbf{P}} & \bar{\mathbf{P}}\mathbf{B} & \mathbf{C}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0.$$

3. *There exists* $\bar{\mathbf{Q}} = \bar{\mathbf{Q}}^\mathsf{T} > 0$ *such that*

$$\mathbf{F}_3 = \begin{bmatrix} \mathbf{A}\bar{\mathbf{Q}} + \bar{\mathbf{Q}}\mathbf{A}^\mathsf{T} & \mathbf{B} & \bar{\mathbf{Q}}\mathbf{C}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0.$$

Proof of $1 \Leftrightarrow 2$

- Write $\mathbf{F}_1 < 0$ as
$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} \\ * & -\gamma^2\mathbf{1} \end{bmatrix} - \begin{bmatrix} \mathbf{C}^\mathsf{T} \\ \mathbf{D}^\mathsf{T} \end{bmatrix} (-\mathbf{1})^{-1} \begin{bmatrix} \mathbf{C} & \mathbf{D} \end{bmatrix} < 0.$$
- Apply the Schur complement lemma, which is an iff statement, to get
$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} & \mathbf{C}^\mathsf{T} \\ * & -\gamma^2\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\mathbf{1} \end{bmatrix} < 0.$$
- A congruence transformation does not alter definiteness, that is, $\mathbf{Q} > 0 \Leftrightarrow \mathbf{TQT}^\mathsf{T} > 0$ for any full rank $\mathbf{T}$.
- For a congruence transformation, consider
$$\mathbf{T} = \operatorname{diag}\left(\gamma^{-\frac{1}{2}}\mathbf{1}, \gamma^{-\frac{1}{2}}\mathbf{1}, \gamma^{\frac{1}{2}}\mathbf{1}\right),$$
where $\mathbf{T}$ is full rank.
- Apply a congruence transformation in $\mathbf{T}$, which is another iff procedure, to get
$$\mathbf{T}\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} & \mathbf{C}^\mathsf{T} \\ * & -\gamma^2\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\mathbf{1} \end{bmatrix}\mathbf{T}^\mathsf{T} < 0, \quad \Leftrightarrow \quad \begin{bmatrix} \bar{\mathbf{P}}\mathbf{A} + \mathbf{A}^\mathsf{T}\bar{\mathbf{P}} & \bar{\mathbf{P}}\mathbf{B} & \mathbf{C}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0,$$
where $\bar{\mathbf{P}} = \mathbf{P}\gamma^{-1}$, and hence $\bar{\mathbf{P}} = \bar{\mathbf{P}}^\mathsf{T} > 0$.

References

These slides are based on [1–3], [4, Ch. 2], [5, Ch. 12], [6, Ch. 12].

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