$\mathcal{H}_2$ Control and Estimation — The LMI Solution —

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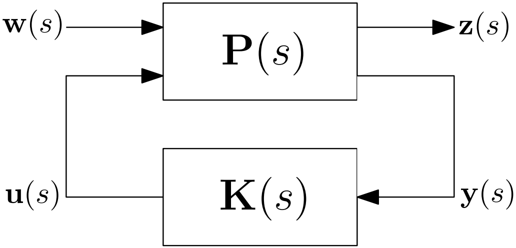
The Standard Control Problem

- Consider a **generalized plant**, $\mathbf{P}(s)$, connected in feedback with a **controller**, $\mathbf{K}(s)$.



- Inputs and outputs are as follows.
  - $\mathbf{w} \in \mathbb{R}^{n_w}$: exogenous inputs.
  - $\mathbf{z} \in \mathbb{R}^{n_z}$: regulated output, also called performance output.
  - $\mathbf{u} \in \mathbb{R}^{n_u}$: control input (controller output [applied by actuators]).
  - $\mathbf{y} \in \mathbb{R}^{n_y}$: plant output (controller input [a function of measurements]).
- Design $\mathbf{K}(s)$ to minimize (in terms of some sort of "cost/objective function") the impact of $\mathbf{w}$ on $\mathbf{z}$.

Static Full-State $\mathcal{H}_2$ Control Design Problem

- In this case, the generalized plant is of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1\mathbf{w}(t) + \mathbf{B}_2\mathbf{u}(t), \qquad \mathbf{x}_0 = \mathbf{x}(0), \tag{1}$$

$$\mathbf{z}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{0}\mathbf{w}(t) + \mathbf{D}_{12}\mathbf{u}(t), \tag{2}$$

$$\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{0}\mathbf{w}(t) + \mathbf{0}\mathbf{u}(t), \tag{3}$$

  so that $\mathbf{D}_{11} = \mathbf{0}$, $\mathbf{C}_2 = \mathbf{1}$, $\mathbf{D}_{21} = \mathbf{0}$, and $\mathbf{D}_{22} = \mathbf{0}$.

- In this case, $\mathbf{y}(t) = \mathbf{x}(t)$ (i.e., all states are measured).
- Consider a controller form

$$\mathbf{y}_K(t) = \mathbf{D}_K\mathbf{u}_K(t),$$
$$\mathbf{u}(t) = -\mathbf{K}\mathbf{y}(t)$$

  where $\mathbf{y}_K(t) = \mathbf{u}(t)$, $\mathbf{u}_K(t) = \mathbf{y}(t) = \mathbf{x}(t)$, and $\mathbf{D}_K = -\mathbf{K}$.

- This is a static full-state feedback controller.
- In this case, negative feedback is assumed. Could just as well pick $\mathbf{D}_K = \mathbf{K}$ and, in many books/papers, this is what is done.

The Closed-Loop System

- The closed-loop system is

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_2\mathbf{K})\mathbf{x}(t) + \mathbf{B}_1\mathbf{w}(t), \qquad \mathbf{x}_0 = \mathbf{x}(0), \tag{4}$$
$$\mathbf{z}(t) = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\mathbf{x}(t), \tag{5}$$

  or

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{zw}\mathbf{x}(t) + \mathbf{B}_{zw}\mathbf{w}(t), \qquad \mathbf{x}_0 = \mathbf{x}(0),$$
$$\mathbf{z}(t) = \mathbf{C}_{zw}\mathbf{x}(t),$$

  where $\mathbf{A}_{zw} = \mathbf{A} - \mathbf{B}_2\mathbf{K}$, $\mathbf{B}_{zw} = \mathbf{B}_1$, and $\mathbf{C}_{zw} = \mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}$.
- The associated closed-loop transfer matrix is

$$\mathbf{T}_{zw}(s) = \mathbf{C}_{zw}(s\mathbf{1} - \mathbf{A}_{zw})^{-1}\mathbf{B}_{zw} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})(s\mathbf{1} - (\mathbf{A} - \mathbf{B}_2\mathbf{K}))^{-1}\mathbf{B}_1.$$

- The closed-loop impulse response matrix is

$$\mathbf{T}_{zw}(t) = \mathbf{C}_{zw}\exp(\mathbf{A}_{zw}t)\mathbf{B}_{zw} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\exp\big((\mathbf{A} - \mathbf{B}_2\mathbf{K})t\big)\mathbf{B}_1.$$

The Optimization Problem

The control design problem at hand is

$$\begin{aligned} \underset{\mathbf{K} \in \mathbb{R}^{n_u \times n_x}}{\text{minimize}} \quad & \|\boldsymbol{\mathcal{T}}_{zw}\|_2 \\ \text{such that} \quad & \text{closed-loop system is asymptotically stable,} \end{aligned}$$

where $\mathbf{z}(t) = (\boldsymbol{\mathcal{T}}_{zw}\mathbf{w})(t)$ and $\mathbf{z}(s) = \mathbf{T}_{zw}(s)\mathbf{w}(s)$.

- Q Why is minimizing the closed-loop $\mathcal{H}_2$ norm a good idea?
- A1 An impulse can be considered a “worst case” disturbance. Find $\mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ to minimize the impact of an impulsive disturbance.
- A2
  - Say the frequency content of the disturbance is unknown.
  - Find $\mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ to minimize $|\mathcal{T}_{zw}(j\omega)|^2$ over $\omega \in \mathbb{R}$.
  - By minimizing $|\mathcal{T}_{zw}(j\omega)|^2$ over $\omega \in \mathbb{R}$, $|\mathcal{T}_{zw}(j\omega)|$ is minimized over $\omega \in \mathbb{R}$, which in turn means minimizing the area under the magnitude portion of the Bode diagram, thereby minimizing the impact of disturbances over all frequencies.
- The solution that uses convex optimization and LMIs will be considered.

Using Convex Optimization and LMIs to Compute the $\mathcal{H}_2$ Norm

Theorem

Consider $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$ *where* $\mathbf{A}$ *is Hurwitz,* $(\mathbf{A}, \mathbf{B})$ *is controllable, and* $(\mathbf{A}, \mathbf{C})$ *is observable. The following conditions are equivalent.*

1. $\|\mathcal{G}\|_2 < \nu$.
2. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\begin{aligned} \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B}\right) &< \nu^2, \\ \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} &< 0. \end{aligned}$$

3. *There exists* $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$ *such that*

$$\begin{aligned} \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}\right) &< \nu^2, \\ \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} &< 0. \end{aligned}$$

Theorem (Schur Complement Lemma)

Let the symmetric matrix $\mathbf{A}$ *be partitioned as*

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^{\mathsf{T}} & \mathbf{A}_{22} \end{bmatrix}.$$

Then,

$$\begin{aligned} \mathbf{A} < 0 \quad &\Leftrightarrow \quad \mathbf{A}_{11} < 0, \quad \mathbf{A}_{22} - \mathbf{A}_{12}^{\mathsf{T}}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} < 0 \\ &\Leftrightarrow \quad \mathbf{A}_{22} < 0, \quad \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\mathsf{T}} < 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{A} > 0 \quad &\Leftrightarrow \quad \mathbf{A}_{11} > 0, \quad \mathbf{A}_{22} - \mathbf{A}_{12}^{\mathsf{T}}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} > 0 \\ &\Leftrightarrow \quad \mathbf{A}_{22} > 0, \quad \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\mathsf{T}} > 0 \end{aligned}$$

Theorem (Congruence Transformation)

Consider $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{W} \in \mathbb{R}^{n\times n}$ *with rank*$(\mathbf{W}) = n$. *Then,*

$$\mathbf{Q} < 0 \quad \Leftrightarrow \quad \mathbf{W}\mathbf{Q}\mathbf{W}^{\mathsf{T}} < 0.$$

Problem ($\mathcal{H}_2$ Control Problem)

Design a static full-state feedback gain $\mathbf{K}$ *such that* $\|\mathcal{T}_{zw}\|_2 < \nu$ *holds for* $0 < \nu < \infty$*.*

The set of symmetric $\mathbb{R}^{n\times n}$ matrices is denoted $\mathbb{S}^{n\times n} = \left\{\mathbf{P} \in \mathbb{R}^{n\times n} \mid \mathbf{P} = \mathbf{P}^{\mathsf{T}}\right\}$.

Theorem

Given $0 < \nu < \infty$*, the* $\mathcal{H}_2$ *control problem has a solution iff there exists* $\mathbf{X} \in \mathbb{S}^{n_x\times n_x}$*,* $\mathbf{Z} \in \mathbb{S}^{n_z\times n_z}$*, and* $\mathbf{F} \in \mathbb{R}^{n_u\times n_x}$ *such that*

$$\mathbf{Z} > 0, \qquad \mathbf{X} > 0, \qquad \mathrm{tr}(\mathbf{Z}) < \nu^2, \tag{6}$$

$$\begin{bmatrix} \mathbf{Z} & (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F}) \\ (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F})^{\mathsf{T}} & \mathbf{X} \end{bmatrix} > 0, \tag{7}$$

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathsf{T}} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} + \mathbf{B}_1\mathbf{B}_1^{\mathsf{T}} < 0. \tag{8}$$

The static full-state feedback gain is given by $\mathbf{K} = \mathbf{F}\mathbf{X}^{-1}$*.*

The design variables are $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{F}$ (but not ν, which is given).

As stated, the above theorem is a *feasibility problem* (i.e., not yet an optimization problem).

Proof

- Recall that the closed-loop transfer matrix is

$$\mathbf{T}_{zw}(s) = \mathbf{C}_{zw}(s\mathbf{1} - \mathbf{A}_{zw})^{-1}\mathbf{B}_{zw} = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})(s\mathbf{1} - (\mathbf{A} - \mathbf{B}_2\mathbf{K}))^{-1}\mathbf{B}_1.$$

- Also recall that $\|\mathcal{T}_{zw}\|_2 < \nu$ iff exists $\mathbf{X} = \mathbf{X}^\mathsf{T} > 0$ such that

$$\begin{aligned} \operatorname{tr}\left(\mathbf{C}_{zw}\mathbf{X}\mathbf{C}_{zw}^\mathsf{T}\right) &< \nu^2, \\ \mathbf{A}_{zw}\mathbf{X} + \mathbf{X}\mathbf{A}_{zw}^\mathsf{T} + \mathbf{B}_{zw}\mathbf{B}_{zw}^\mathsf{T} &< 0. \end{aligned}$$

Proof Continued

- Using the “slack variable” $\mathbf{Z} = \mathbf{Z}^\mathsf{T} > 0$, note that

$$
\begin{aligned}
& \operatorname{tr}\left(\mathbf{C}_{zw}\mathbf{X}\mathbf{C}_{zw}^\mathsf{T}\right) < \nu^2 \\
\Leftrightarrow \quad & \mathbf{C}_{zw}\mathbf{X}\mathbf{C}_{zw}^\mathsf{T} < \mathbf{Z}, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \quad \text{(Not an obvious step.)} \\
\Leftrightarrow \quad & \mathbf{Z} - \mathbf{C}_{zw}\mathbf{X}\mathbf{X}^{-1}\mathbf{X}\mathbf{C}_{zw}^\mathsf{T} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{Z} & \mathbf{C}_{zw}\mathbf{X} \\ \mathbf{X}\mathbf{C}_{zw}^\mathsf{T} & \mathbf{X} \end{bmatrix} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \quad \text{(Schur complement used.)} \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{Z} & (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F}) \\ (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F})^\mathsf{T} & \mathbf{X} \end{bmatrix} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2,
\end{aligned}
$$

where the Schur complement has been used and $\mathbf{F} = \mathbf{K}\mathbf{X}$.

Proof Continued

- In a similar fashion,

$$\begin{aligned} \mathbf{A}_{zw}\mathbf{X} + \mathbf{X}\mathbf{A}_{zw}^{\mathsf{T}} + \mathbf{B}_{zw}\mathbf{B}_{zw}^{\mathsf{T}} &< 0, \\ \Leftrightarrow \quad (\mathbf{A} - \mathbf{B}_2\mathbf{K})\mathbf{X} + \mathbf{X}(\mathbf{A} - \mathbf{B}_2\mathbf{K})^{\mathsf{T}} + \mathbf{B}_1\mathbf{B}_1^{\mathsf{T}} &< 0, \\ \Leftrightarrow \quad \mathbf{A}\mathbf{X} - \mathbf{B}_2\mathbf{K}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathsf{T}} - \mathbf{X}\mathbf{K}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} + \mathbf{B}_1\mathbf{B}_1^{\mathsf{T}} &< 0, \\ \Leftrightarrow \quad \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathsf{T}} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} + \mathbf{B}_1\mathbf{B}_1^{\mathsf{T}} &< 0, \end{aligned}$$

where $\mathbf{F} = \mathbf{K}\mathbf{X}$.

Comments

Q Why do we need the slack variable $\mathbf{Z}$ and the Schur complement?

A We can't work directly with

$$\operatorname{tr}\left(\mathbf{C}_{zw}\mathbf{X}\mathbf{C}_{zw}^{\mathsf{T}}\right) < \nu^2,$$
$$\mathbf{A}_{zw}\mathbf{X} + \mathbf{X}\mathbf{A}_{zw}^{\mathsf{T}} + \mathbf{B}_{zw}\mathbf{B}_{zw}^{\mathsf{T}} < 0,$$

because these inequalities are not convex in $\mathbf{X}$ and $\mathbf{K}$, that is, not LMIs in $\mathbf{X}$ and $\mathbf{K}$.

- We need to use the change of variables $\mathbf{F} = \mathbf{K}\mathbf{X}$ and the slack variable (plus the Schur complement) to convexify/simplify the problem.

Optimal $\mathcal{H}_2$ Controller

The smallest ν such that $\|\mathcal{T}_{zw}\|_2 < \nu$ can be found by solve the following convex optimization problem subject to LMI constraints:

$$\begin{aligned} \underset{\mathbf{Z}=\mathbf{Z}^\mathsf{T},\ \mathbf{X}=\mathbf{X}^\mathsf{T},\ \mathbf{F},\ \nu}{\text{minimize}} \quad & \nu^2 \\ \text{such that} \quad & \nu > 0, \quad \text{(6), (7), and (8).} \end{aligned}$$

The optimal static state-feedback gain (in an $\mathcal{H}_2$ sense) is then $\mathbf{K} = \mathbf{F}\mathbf{X}^{-1}$.

- The above is an *optimization problem*. The design (optimization) variables are $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{F}$, and ν.
- Note, the word "optimal" is meaningless without specifying "with respect to what" (i.e., the form of the cost function). In this case the $\mathbf{K}$ is optimal in an $\mathcal{H}_2$ sense.

The $\mathcal{H}_2$ Estimation Problem

- Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \boldsymbol{\mathcal{B}}_1\mathbf{w}(t) + \mathbf{B}_2\mathbf{u}(t), \qquad \mathbf{x}_0 = \mathbf{x}(0),\\ \mathbf{y}(t) &= \boldsymbol{\mathcal{C}}_2\mathbf{x}(t) + \boldsymbol{\mathcal{D}}_{21}\mathbf{w}(t)\end{aligned}$$

  where $\mathbf{w}$ can be decomposed into process noise (i.e., a disturbance) and measurement noise.

- Consider an observer of the form

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_2\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t))\\ &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_2\mathbf{u}(t) + \mathbf{L}(\boldsymbol{\mathcal{C}}_2\mathbf{x}(t) + \boldsymbol{\mathcal{D}}_{21}\mathbf{w}(t) - \boldsymbol{\mathcal{C}}_2\hat{\mathbf{x}}(t))\\ &= (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2)\hat{\mathbf{x}}(t) + \mathbf{B}_2\mathbf{u}(t) + \mathbf{L}\boldsymbol{\mathcal{C}}_2\mathbf{x}(t) + \mathbf{L}\boldsymbol{\mathcal{D}}_{21}\mathbf{w}(t).\end{aligned}$$

- Next, consider the error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, and the error dynamics

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\\ &= \mathbf{A}\mathbf{x}(t) + \boldsymbol{\mathcal{B}}_1\mathbf{w}(t) + \mathbf{B}_2\mathbf{u}(t) - (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2)\hat{\mathbf{x}}(t) - \mathbf{B}_2\mathbf{u}(t) - \mathbf{L}\boldsymbol{\mathcal{C}}_2\mathbf{x}(t) - \mathbf{L}\boldsymbol{\mathcal{D}}_{21}\mathbf{w}(t)\\ &= (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2)\mathbf{e}(t) + (\boldsymbol{\mathcal{B}}_1 - \mathbf{L}\boldsymbol{\mathcal{D}}_{21})\mathbf{w}(t)\end{aligned}$$

- Let the performance be $\mathbf{z}(t) = \boldsymbol{\mathcal{C}}_1\mathbf{e}(t)$.

- The transfer matrix $\mathbf{T}_{zw}(s)$ is

$$\mathbf{T}_{zw}(s) = \mathbf{C}_{zw}(s\mathbf{1} - \mathbf{A}_{zw})^{-1}\mathbf{B}_{zw} = \boldsymbol{\mathcal{C}}_1(s\mathbf{1} - (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2))^{-1}(\boldsymbol{\mathcal{B}}_1 - \mathbf{L}\boldsymbol{\mathcal{D}}_{21}),$$

while the impulse response matrix is

$$\mathbf{T}_{zw}(t) = \mathbf{C}_{zw}\exp(\mathbf{A}_{zw}t)\mathbf{B}_{zw} = \boldsymbol{\mathcal{C}}_1 \exp\big((\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2)t\big)(\boldsymbol{\mathcal{B}}_1 - \mathbf{L}\boldsymbol{\mathcal{D}}_{21}).$$

Problem ($\mathcal{H}_2$ Estimation Problem)

Design an observer gain $\mathbf{L}$ *such that* $\|\boldsymbol{\mathcal{T}}_{zw}\|_2 < \nu$ *holds for* $0 < \nu < \infty$.

Theorem

Given $0 < \nu < \infty$, the $\mathcal{H}_2$ estimation problem has a solution iff there exists $\mathbf{X} \in \mathbb{S}^{n_x \times n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_w \times n_w}$, and $\mathbf{F} \in \mathbb{R}^{n_y \times n_x}$ such that

$$\mathbf{Z} > 0, \qquad \mathbf{X} > 0, \qquad \mathrm{tr}(\mathbf{Z}) < \nu^2, \tag{9}$$

$$\begin{bmatrix} \mathbf{Z} & (\mathbf{X}\boldsymbol{\mathcal{B}}_1 - \mathbf{F}\boldsymbol{\mathcal{D}}_{21})^\mathsf{T} \\ (\mathbf{X}\boldsymbol{\mathcal{B}}_1 - \mathbf{F}\boldsymbol{\mathcal{D}}_{21}) & \mathbf{X} \end{bmatrix} > 0, \tag{10}$$

$$\mathbf{X}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X} - \mathbf{F}\boldsymbol{\mathcal{C}}_2 - \boldsymbol{\mathcal{C}}_2^\mathsf{T}\mathbf{F}^\mathsf{T} + \boldsymbol{\mathcal{C}}_1^\mathsf{T}\boldsymbol{\mathcal{C}}_1 < 0. \tag{11}$$

The observer gain is given by $\mathbf{L} = \mathbf{X}^{-1}\mathbf{F}$.

- The design variables are $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{F}$ (but not ν, which is given).
- As stated, the above theorem is a *feasibility problem* (i.e., not yet an optimization problem).

Proof

- Recall that the closed-loop transfer matrix is

$$\mathbf{T}_{zw}(s) = \mathbf{C}_{zw}(s\mathbf{1} - \mathbf{A}_{zw})^{-1}\mathbf{B}_{zw} = \boldsymbol{\mathcal{C}}_1(s\mathbf{1} - (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2))^{-1}(\boldsymbol{\mathcal{B}}_1 - \mathbf{L}\boldsymbol{\mathcal{D}}_{21}).$$

- Also recall that $\|\boldsymbol{\mathcal{T}}_{zw}\|_2 < \nu$ iff exists $\mathbf{X} = \mathbf{X}^\mathsf{T} > 0$ such that

$$\operatorname{tr}\left(\mathbf{B}_{zw}^\mathsf{T}\mathbf{X}\mathbf{B}_{zw}\right) < \nu^2,$$
$$\mathbf{X}\mathbf{A}_{zw} + \mathbf{A}_{zw}^\mathsf{T}\mathbf{X} + \mathbf{C}_{zw}^\mathsf{T}\mathbf{C}_{zw} < 0.$$

- Using the “slack variable” $\mathbf{Z} = \mathbf{Z}^\mathsf{T} > 0$, note that

$$\begin{aligned}
& \operatorname{tr}\left(\mathbf{B}_{zw}^\mathsf{T}\mathbf{X}\mathbf{B}_{zw}\right) < \nu^2 \\
\Leftrightarrow \quad & \mathbf{B}_{zw}^\mathsf{T}\mathbf{X}\mathbf{B}_{zw} < \mathbf{Z}, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \quad \text{(Not an obvious step.)} \\
\Leftrightarrow \quad & \mathbf{Z} - \mathbf{B}_{zw}^\mathsf{T}\mathbf{X}\mathbf{X}^{-1}\mathbf{X}\mathbf{B}_{zw} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{Z} & \mathbf{B}_{zw}^\mathsf{T}\mathbf{X} \\ \mathbf{X}\mathbf{B}_{zw} & \mathbf{X} \end{bmatrix} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2, \quad \text{(Schur complement used.)} \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{Z} & (\mathbf{X}\boldsymbol{\mathcal{B}}_1 - \mathbf{F}\boldsymbol{\mathcal{D}}_{21})^\mathsf{T} \\ (\mathbf{X}\boldsymbol{\mathcal{B}}_1 - \mathbf{F}\boldsymbol{\mathcal{D}}_{21}) & \mathbf{X} \end{bmatrix} > 0, \quad \operatorname{tr}(\mathbf{Z}) < \nu^2,
\end{aligned}$$

where the Schur complement has been used and $\mathbf{F} = \mathbf{X}\mathbf{L}$.

Proof Continued

- In a similar fashion,

$$
\begin{aligned}
\mathbf{X}\mathbf{A}_{zw} + \mathbf{A}_{zw}^{\mathsf{T}}\mathbf{X} + \mathbf{C}_{zw}^{\mathsf{T}}\mathbf{C}_{zw} &< 0, \\
\Leftrightarrow \quad \mathbf{X}(\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2) + (\mathbf{A} - \mathbf{L}\boldsymbol{\mathcal{C}}_2)^{\mathsf{T}}\mathbf{X} + \boldsymbol{\mathcal{C}}_1^{\mathsf{T}}\boldsymbol{\mathcal{C}}_1 &< 0, \\
\Leftrightarrow \quad \mathbf{X}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{X} - \mathbf{F}\boldsymbol{\mathcal{C}}_2 - \boldsymbol{\mathcal{C}}_2^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} + \boldsymbol{\mathcal{C}}_1^{\mathsf{T}}\boldsymbol{\mathcal{C}}_1 &< 0,
\end{aligned}
$$

where $\mathbf{F} = \mathbf{X}\mathbf{L}$.

Optimal $\mathcal{H}_2$ Estimator

The smallest ν such that $\|\mathcal{T}_{zw}\|_2 < \nu$ can be found by solve the following convex optimization problem subject to LMI constraints:

$$\begin{array}{ll} \underset{\mathbf{Z}=\mathbf{Z}^\mathsf{T},\ \mathbf{X}=\mathbf{X}^\mathsf{T},\ \mathbf{F},\ \nu}{\text{minimize}} & \nu^2 \\ \text{such that} & \nu > 0, \ \ \text{(9), (10), and (11).} \end{array}$$

The optimal observer gain (in an $\mathcal{H}_2$ sense) is then $\mathbf{L} = \mathbf{X}^{-1}\mathbf{F}$.

- The above is an *optimization problem*. The design (optimization) variables are $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{F}$, and ν.
- Note, the word “optimal” is meaningless without specifying “with respect to what” (i.e., the form of the cost function). In this case the $\mathbf{L}$ is optimal in an $\mathcal{H}_2$ sense.

Convex Optimization + LMIs versus Analytic Solution

Q Why use the convex optimization + LMIs solution(s) versus an analytic solution (i.e., the ARE)?

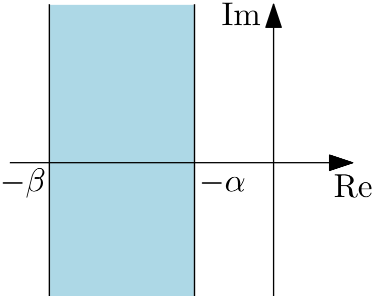
A Because additional constraints can be easily incorporated into the controller synthesis problem.

Theorem

Let $\mathcal{R} = \{\lambda \in \mathbb{C} \mid -\beta < Re\{\lambda\} < -\alpha\}$ *be a strip region in the complex plane. The eigenvalues of* $\mathbf{A}$ *lie in* $\mathcal{R}$ *(i.e.,* $\lambda_i(\mathbf{A}) \in \mathcal{R},\ i = 1, 2, \ldots, n$*) iff there exists* $\mathbf{P} \in \mathbb{S}^{n \times n}$ *such that* $\mathbf{P} > 0$*,*

$$\mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + 2\alpha\mathbf{P} < 0,$$

$$\mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + 2\beta\mathbf{P} > 0.$$



Problem ($\mathcal{H}_2$ Control Problem with Strip Region Constraint)

Design a static full-state feedback gain $\mathbf{K}$ *such that* $\|\mathcal{T}_{zw}\|_2 < \nu$ *holds for* $0 < \nu < \infty$ *and the closed-loop eigenvalues are located in the strip region described by* $\mathcal{R} = \{\lambda \in \mathbb{C} \mid -\beta < Re\{\lambda\} < -\alpha\}$.

Theorem

Given $0 < \nu < \infty$, *the* $\mathcal{H}_2$ *control problem with strip region constraint has a solution if there exists* $\mathbf{X} \in \mathbb{S}^{n_x \times n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z \times n_z}$, *and* $\mathbf{F} \in \mathbb{R}^{n_u \times n_x}$ *such that*

$$\mathbf{Z} > 0, \qquad \mathbf{X} > 0, \qquad \mathrm{tr}(\mathbf{Z}) < \nu^2, \tag{12}$$

$$\begin{bmatrix} \mathbf{Z} & (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F}) \\ (\mathbf{C}_1\mathbf{X} - \mathbf{D}_{12}\mathbf{F})^\mathsf{T} & \mathbf{X} \end{bmatrix} > 0, \tag{13}$$

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\mathsf{T} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^\mathsf{T}\mathbf{B}_2^\mathsf{T} + \mathbf{B}_1\mathbf{B}_1^\mathsf{T} < 0, \tag{14}$$

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\mathsf{T} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^\mathsf{T}\mathbf{B}_2^\mathsf{T} + 2\alpha\mathbf{X} < 0, \tag{15}$$

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\mathsf{T} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^\mathsf{T}\mathbf{B}_2^\mathsf{T} + 2\beta\mathbf{X} > 0. \tag{16}$$

The static full-state feedback gain is given by $\mathbf{K} = \mathbf{F}\mathbf{X}^{-1}$.

References

These slides are based on [1, 2].

[1] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach*. New York, NY: Springer, Texts in Applied Mathematics, 2000.

[2] G.-R. Duan and H.-H. Yu, *LMIs in Control Systems: Analysis, Design and Applications*. Boca Raton, FL: Routledge, Taylor and Francis Group, CRC Press, 2013.