Controllability, Stabilizability, and Static Full-State Feedback Control

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- Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{1}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \tag{2}$$

- Assume $\mathbf{C} = \mathbf{1}$ and $\mathbf{D} = \mathbf{0}$.
  - Is this realistic? No ...
  - A sensor for every state would be needed.
  - If you have 1000 states, are you going to have 1000 sensors? No, you're not.
- Consider control of the form

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t).$$

- This is called *static full-state feedback control* because the whole state is assumed available for control purposes, and the control gain is constant (i.e., "static").
- $\mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ is the static full-state feedback control gain.

- Using (1), the closed-loop dynamics are

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t),\end{aligned}$$

  where $\mathbf{x}_0 = \mathbf{x}(0)$.

- The closed-loop response, as well as closed-loop stability properties, are dictated by the eigenvalues of

$$\mathbf{A} - \mathbf{B}\mathbf{K}.$$

Q. Where should the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ be placed to realize closed-loop AS of the EP $\bar{\mathbf{x}} = \mathbf{0}$?

A. In the OLHP! That is, design $\mathbf{K}$ such that $\mathrm{Re}\,\{\lambda_i(\mathbf{A} - \mathbf{B}\mathbf{K})\} < 0$, $i = 1, 2, \ldots, n_x$!

- But wait . . .
- Is it always possible to place the eigenvalues of $\mathbf{A} - \mathbf{BK}$ in the OLHP?
- Let's look at some examples.

Example

- Consider the process model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{3}$$

- Let the control be

$$u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{4}$$

- Then, the closed-loop dynamics are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} && (5) \\ &= \begin{bmatrix} 0 & 1 \\ -k_1 & 1 - k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. && (6) \end{aligned}$$

- The eigenvalues of $\mathbf{A} - \mathbf{BK}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda\mathbf{1} - (\mathbf{A} - \mathbf{BK})\right) && (7)\\
&= \det\begin{bmatrix} \lambda & -1 \\ k_1 & \lambda - 1 + k_2 \end{bmatrix} && (8)\\
&= \lambda(\lambda - 1 + k_2) + k_1 && (9)\\
&= \lambda^2 + \lambda(-1 + k_2) + k_1. && (10)
\end{aligned}
$$

- Recall that the roots of the second-order polynomial $a_2\lambda^2 + a_1\lambda + a_0 = 0$ are strictly in the OLHP iff $a_i > 0$, $i = 1, 2, 3$.
- Thus, pick $k_1 > 0$ and $k_2 > 1$ and the eigenvalues of $\mathbf{A} - \mathbf{BK}$ will be in the OLHP! Great!

- Say the desired eigenvalues of $\mathbf{A} - \mathbf{BK}$ are the roots of $\alpha(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$.
- Pick $k_1 = 2$ and $k_2 = 4$.
- This is *eigenvalue placement*. In classical control (i.e., MECH 412) this is called *pole placement*.

Example

- Consider the process model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \tag{11}$$

- As before, let the control be

$$u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{12}$$

- Then, the closed-loop dynamics are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} & (13) \\ &= \begin{bmatrix} -k_1 & 1 - k_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. & (14) \end{aligned}$$

- The eigenvalues of $\mathbf{A} - \mathbf{BK}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda \mathbf{1} - (\mathbf{A} - \mathbf{BK})\right) && (15)\\
&= \det\begin{bmatrix} \lambda + k_1 & -1 + k_2 \\ 0 & \lambda - 1 \end{bmatrix} && (16)\\
&= (\lambda + k_1)(\lambda - 1). && (17)
\end{aligned}
$$

- Although one eigenvalue can be placed at $\lambda_1 = -k_1$, the other eigenvalue is $\lambda_2 = 1$ no matter what!
- Thus, the EP $\bar{\mathbf{x}} = \mathbf{0}$ of the closed-loop system cannot be rendered AS. Yikes!

Example

- Consider the process model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \tag{18}$$

- As before, let the control be

$$u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{19}$$

- Then, the closed-loop dynamics are

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} -k_1 & 1-k_2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{21}$$

- The eigenvalues of $\mathbf{A} - \mathbf{BK}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda \mathbf{1} - (\mathbf{A} - \mathbf{BK})\right) && (22)\\
&= \det \begin{bmatrix} \lambda + k_1 & -1 + k_2 \\ 0 & \lambda + 1 \end{bmatrix} && (23)\\
&= (\lambda + k_1)(\lambda + 1). && (24)
\end{aligned}
$$

- Although one eigenvalue can be placed at $\lambda_1 = -k_1$, the other eigenvalue is $\lambda_2 = -1$ no matter what.
- Thus, the EP $\bar{\mathbf{x}} = \mathbf{0}$ can be rendered AS, but the eigenvalues of $\mathbf{A} - \mathbf{BK}$ cannot be arbitrary placed in the OLHP.

Controllability

- The previous examples indicate that some $(\mathbf{A}, \mathbf{B})$ pairs allow for the eigenvalues of $\mathbf{A} - \mathbf{BK}$ to be
  - arbitrarily placed in the OLHP, or
  - partially placed in the OLHP,

  via proper design of $\mathbf{K}$.
- The case of arbitrary eigenvalue placement leads to the definition of *controllability*.

Definition

$(\mathbf{A}, \mathbf{B})$ is *controllable* if the eigenvalues of $\mathbf{A} - \mathbf{BK}$ can be arbitrarily assigned, with complex eigenvalues appearing in complex-conjugate pairs, by a suitable choice of $\mathbf{K}$.

- There are many ways to assess if $(\mathbf{A}, \mathbf{B})$ is controllable.

Theorem

The following are equivalent.

- $(\mathbf{A}, \mathbf{B})$ *is controllable.*
- *The controllability matrix*

$$\mathbf{Q}_\mathrm{c} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n_x - 1}\mathbf{B} \end{bmatrix} \tag{25}$$

  is full-row rank, $\mathrm{rank}\mathbf{Q}_\mathrm{c} = n_x$.
- *The controllability Gramian*

$$\mathbf{W}_\mathrm{c}(t) = \int_0^t \exp(\mathbf{A}\tau)\mathbf{B}\mathbf{B}^\mathsf{T} \exp(\mathbf{A}^\mathsf{T}\tau)\,\mathrm{d}\tau \tag{26}$$

  is positive definite $\forall t > 0$.
- *The matrix* $\begin{bmatrix} \mathbf{A} - \lambda\mathbf{1} & \mathbf{B} \end{bmatrix}$ *has full-row rank* $\forall \lambda \in \mathbb{C}$.
- *Let* λ *and* $\mathbf{w}$ *be any left eigenpair of* $\mathbf{A}$ *(i.e.,* $\mathbf{w}^\mathsf{H}\mathbf{A} = \lambda^\mathsf{H}\mathbf{w}^\mathsf{H}$*). Then* $\mathbf{w}^\mathsf{H}\mathbf{B} \neq \mathbf{0}$.

In practice, use

```
Qc = control.ctrb(A, B)

Wc = control.gram(P, 'c')
```

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{27}$$

- The controllability matrix is

$$\mathbf{Q}_{\mathrm{c}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{28}$$

- $\mathrm{rank}\mathbf{Q}_{\mathrm{c}} = 2 = n_x$.
- Thus, the system is controllable, meaning the eigenvalues of $\mathbf{A} - \mathbf{BK}$ can be assigned arbitrarily.
- This result jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{29}$$

- The controllability matrix is

$$\mathbf{Q}_\mathrm{c} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{30}$$

- $\mathrm{rank}\mathbf{Q}_\mathrm{c} = 1 \neq n_x = 2$.
- Thus, the system is not controllable, meaning the eigenvalues of $\mathbf{A} - \mathbf{BK}$ cannot be assigned arbitrarily.
- This result also jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{31}$$

- The controllability matrix is

$$\mathbf{Q}_{\mathrm{c}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{32}$$

- $\mathrm{rank}\mathbf{Q}_{\mathrm{c}} = 1 \neq n_x = 2.$
- Thus, the system is not controllable, meaning the eigenvalues of $\mathbf{A} - \mathbf{BK}$ cannot be assigned arbitrarily.
- This result, again, jives with our previous findings.

Stabilizability

- The case of partial eigenvalue placement where the eigenvalues of $\mathbf{A} - \mathbf{BK}$ are in the OLHP, but not all eigenvalues are arbitrarily placed, leads to the definition of *stabilizability*.

Definition

$(\mathbf{A}, \mathbf{B})$ is *stabilizable* if there exists $\mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ such that $\mathrm{Re}\left\{\lambda_i(\mathbf{A} - \mathbf{BK})\right\} < 0$, $i = 1, 2, \ldots, n_x$.

- There are many ways to assess if $(\mathbf{A}, \mathbf{B})$ is stabilizable.

Theorem

The following are equivalent.

- $(\mathbf{A}, \mathbf{B})$ *is stabilizable.*
- *The matrix* $\begin{bmatrix} \mathbf{A} - \lambda \mathbf{1} & \mathbf{B} \end{bmatrix}$ *has full-row rank* $\forall \mathrm{Re}\left\{\lambda\right\} \geq 0$.
- *For all* λ *and* $\mathbf{w}$ *such that* $\mathbf{w}^{\mathsf{H}} \mathbf{A} = \lambda^{\mathsf{H}} \mathbf{w}^{\mathsf{H}}$ *and* $\mathrm{Re}\left\{\lambda\right\} \geq 0$, $\mathbf{w}^{\mathsf{H}} \mathbf{B} \neq \mathbf{0}$.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{33}$$

- The $\mathbf{A}$ matrix eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$.
- For $\lambda_1 = 0$, the matrix

$$\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{1} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 1 \\ 0 & 1 - \lambda_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{34}$$

has row-rank equal to 2 (i.e., is full-row rank).

- For $\lambda_2 = 1$, the matrix

$$\begin{bmatrix} \mathbf{A} - \lambda_2 \mathbf{1} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} -\lambda_2 & 1 & 1 \\ 0 & 1 - \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

  has row-rank equal to 1 (i.e., is *not* full-row rank).
- Thus, the system is *not* stabilizable.
- Again, this result jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{36}$$

- The $\mathbf{A}$ matrix eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$.
- For $\lambda_1 = 0$, the matrix

$$\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{1} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 1 \\ 0 & -1-\lambda_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{37}$$

  has row-rank equal to 2 (i.e., is full-row rank).
- Thus, the system is stabilizable, there exists a $\mathbf{K}$ such that all the eigenvalues of $\mathbf{A} - \mathbf{BK}$ will be in the OLHP.
- Again, the jive is real ...

Static Full-State Feedback Control Example

- Consider the plant

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{38}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t). \tag{39}$$

- Notice that $\mathbf{y}(t) = \mathbf{x}(t)$.

Q. Design a static full-state feedback controller so that with the slowest dominant eigenvalue of the closed-loop system is at -1.

A. First, using `control.ctrb`, check the pair $(\mathbf{A}, \mathbf{B})$ is controllable, which is true.

- Next, design the static full-state feedback gain $\mathbf{K}$ using `control.place`.
  - Place the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ at $-1, -2$ so that the slowest dominant closed-loop eigenvalue is -1.

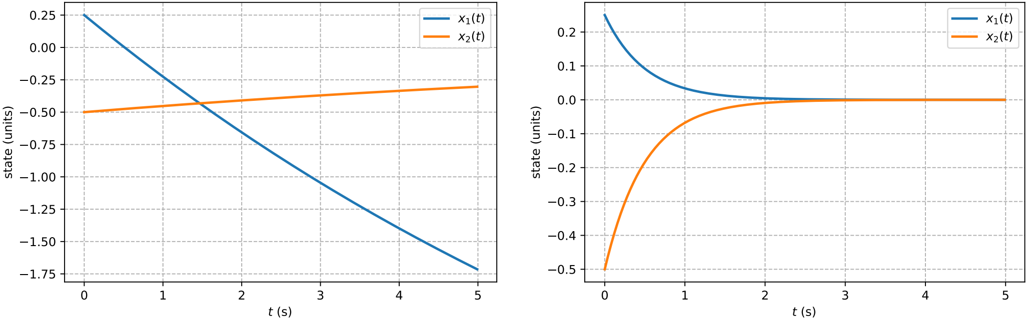

Figure 1: (Left) Open-loop (no control) response from IC. (Right) Closed-loop response from IC using static full-state feedback.

```python
import numpy as np
import control

# Plant
A = np.array([[0, 1],
              [0, -0.1]])
B = np.array([[0],
               [1]])
C = np.array([[1, 0],
               [0, 1]])
D = np.array([[0],
               [0]])
x0 = np.array([0.25, -0.5])  # Initial conditions
n_x, n_u, n_y = A.shape[0], B.shape[1], C.shape[0]  # dimensions
P = control.ss(A, B, C, D)

# Full-state feedback gain design
desired_eigs_K = np.array([-1, -2])
K = control.place(A, B, desired_eigs_K)

```

Full-State Feedback, Coordinate Transformations, and Stabilizability

Took out 2025/01/05

- Consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{u}(t) &= -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t),\end{aligned}$$

  where $\mathbf{G}$ is square and full rank.
- Assume $(\mathbf{A}, \mathbf{B})$ is stabilizable.
- It follows that

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t).\end{aligned}$$

- Using the coordinate transformation $\mathbf{z}(t) = \mathbf{T}\mathbf{x}(t)$ where $\mathbf{T}$ is square and full rank,

$$\dot{\mathbf{z}}(t) = \mathbf{T}^{-1}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{T}\mathbf{z}(t) + \mathbf{T}^{-1}\mathbf{B}\mathbf{G}\mathbf{r}(t).$$

- Here's the critical question: did the state feedback, or the coordinate transformation, alter the stabilizablilty properties?

Theorem

Let $\mathbf{G}$ and $\mathbf{T}$ be any square, full rank, matrices of appropriate dimension. The following are equivalent.

- $(\mathbf{A}, \mathbf{B})$ *is stabilizable.*
- $(\mathbf{T}^{-1}(\mathbf{A} - \mathbf{BK})\mathbf{T}, \mathbf{T}^{-1}\mathbf{BG})$ *is stabilizable.*

- Identify the propositions.
  - $p = \big($ $(\mathbf{A}, \mathbf{B})$ is stabilizable $\big)$.
  - $q = \big($ $(\mathbf{T}^{-1}(\mathbf{A} - \mathbf{BK})\mathbf{T}, \mathbf{T}^{-1}\mathbf{BG})$ is stabilizable $\big)$.

Proof.

- First recall that pre- or post-multiplication of matrix by a full rank matrix does not alter rank.
  - For instance, say $\mathbf{Q} \in \mathbb{R}^{m \times n}$ is full row rank, $\mathbf{P} \in \mathbb{R}^{m \times m}$ is full rank, and $\mathbf{R} \in \mathbb{R}^{n \times n}$ is full rank.
  - Then $\mathrm{rank}\mathbf{Q} = \mathrm{rank}(\mathbf{PQ}) = \mathrm{rank}(\mathbf{QR}) = (\mathbf{PQR})$.

- Next, starting with proposition p, $(\mathbf{A}, \mathbf{B})$ stabilizable means the matrix

$$\begin{bmatrix}\mathbf{A} - \lambda\mathbf{1} & \mathbf{B}\end{bmatrix}$$

has full-row rank $\forall \mathrm{Re}\{\lambda\} \geq 0$.

- Pre- and post-multiplying by $\mathbf{T}^{-1}$ and $\begin{bmatrix}\mathbf{T} & \mathbf{0} \\ -\mathbf{KT} & \mathbf{G}\end{bmatrix}$, respectively, both of which are full rank, gives

$$\mathbf{T}^{-1}\begin{bmatrix}\mathbf{A} - \lambda\mathbf{1} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{T} & \mathbf{0} \\ -\mathbf{KT} & \mathbf{G}\end{bmatrix}.$$

- Recall that pre- or post-multiplication of matrix by a full rank matrix does not alter rank. Thus, the matrix

$$\begin{bmatrix}\mathbf{A} - \lambda\mathbf{1} & \mathbf{B}\end{bmatrix}$$

has full-row rank $\forall \mathrm{Re}\{\lambda\} \geq 0$ iff

$$\mathbf{T}^{-1}\begin{bmatrix}\mathbf{A} - \lambda\mathbf{1} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{T} & \mathbf{0} \\ -\mathbf{KT} & \mathbf{G}\end{bmatrix}$$

has full-row rank $\forall \mathrm{Re}\{\lambda\} \geq 0$.

- Next, notice that

$$\begin{aligned}\mathbf{T}^{-1}\begin{bmatrix}\mathbf{A}-\lambda\mathbf{1} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{T} & \mathbf{0}\\ \mathbf{KT} & \mathbf{G}\end{bmatrix} &= \begin{bmatrix}\mathbf{T}^{-1}\mathbf{A}-\lambda\mathbf{T}^{-1} & \mathbf{T}^{-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{T} & \mathbf{0}\\ \mathbf{KT} & \mathbf{G}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{T}^{-1}\mathbf{AT}-\lambda\mathbf{1}+\mathbf{T}^{-1}\mathbf{BKT} & \mathbf{T}^{-1}\mathbf{BG}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{T}^{-1}(\mathbf{A}-\mathbf{BK})\mathbf{T}-\lambda\mathbf{1} & \mathbf{T}^{-1}\mathbf{BG}\end{bmatrix}.\end{aligned}$$

- Thus, the matrix

$$\begin{bmatrix}\mathbf{A}-\lambda\mathbf{1} & \mathbf{B}\end{bmatrix}$$

has full-row rank $\forall \mathrm{Re}\left\{\lambda\right\} \geq 0$ iff

$$\begin{bmatrix}\mathbf{T}^{-1}(\mathbf{A}-\mathbf{BK})\mathbf{T}-\lambda\mathbf{1} & \mathbf{T}^{-1}\mathbf{BG}\end{bmatrix}$$

has full-row rank $\forall \mathrm{Re}\left\{\lambda\right\} \geq 0$, which is $p \Leftrightarrow q$.

References

These slides are based on [1–3].

[1] R. L. Williams and D. A. Lawrence, *Linear State-Space Control Systems*. Hoboken, NJ: John Wiley & Sons, Inc., 2007.
[2] J. Hespanha, *Linear Systems Theory*. Princeton, NJ: Princeton University Press, 2009.
[3] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, NJ: Prentice Hall, 1998.