Observability, Detectability, and Observers

Prof. James Richard Forbes ©

McGill University, Department of Mechanical Engineering

McGill

- Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \tag{2}$$

- Now $\mathbf{C} \neq \mathbf{1}$ and $\mathbf{D} \neq \mathbf{0}$ where $n_y < n_x$ meaning $\mathbf{C}$ is a wide matrix.
  - Now, this *is* realistic!
  - Any real system will have a limited number of sensors that will measure some subset, or some linear combination, of states.
- With only $\mathbf{y}(t)$ available, and not $\mathbf{x}(t)$, static full-state feedback control of the form

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$$

  is not possible.
- Perhaps the state $\mathbf{x}(t)$ could be *estimated* enabling control of the form

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t),$$

  where $\hat{\mathbf{x}}(t)$ is the *estimated state*.
- Let's investigate estimating the state . . .

Q If $\mathbf{x}(t)$ cannot be measured directly, is it possible to generate an **estimate** of $\mathbf{x}(t)$, call it $\hat{\mathbf{x}}(t)$, such that $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$ where $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$?

A1 Let's "copy" the process model, and look at the error dynamics.

$$\text{Copy of process model:} \quad \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0), \tag{3}$$

$$\begin{aligned} \text{Error dynamics:} \quad \dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - (\mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t)) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\ &= \mathbf{A}\mathbf{e}(t), \quad \mathbf{e}_0 = \mathbf{e}(0) = \mathbf{x}_0 - \hat{\mathbf{x}}_0 \end{aligned} \tag{4}$$

- Thus, $\mathbf{e}(t) = \exp(\mathbf{A}t)\mathbf{e}_0$, and $\bar{\mathbf{e}} = \mathbf{0}$ is an AS EP iff $\mathrm{Re}\{\lambda_i(\mathbf{A})\} < 0$, $i = 1, 2, \ldots, n$, meaning that $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$.
- But what if $\mathrm{Re}\{\lambda_i(\mathbf{A})\} \leq 0$ for one or more i, or even worse, $\mathrm{Re}\{\lambda_i(\mathbf{A})\} > 0$ for just one i?
- Clearly (3) is not a very good **observer**.
  - The reason is that (3) is an **open-loop** observer.

A2 Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

and the observer dynamics

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{\Phi}\hat{\mathbf{x}}(t) + \mathbf{\Gamma}\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0).$$

- What must $\mathbf{\Phi}$, $\mathbf{\Gamma}$, and $\mathbf{L}$ be in order to have $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$ where $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$?
- The error dynamics are

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - (\mathbf{\Phi}\hat{\mathbf{x}}(t) + \mathbf{\Gamma}\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - (\mathbf{\Phi}\hat{\mathbf{x}}(t) + \mathbf{\Gamma}\mathbf{u}(t) + \mathbf{L}\mathbf{C}\mathbf{x}(t) + \mathbf{L}\mathbf{D}\mathbf{u}(t)) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{x}(t) + (\mathbf{B} - \mathbf{L}\mathbf{D} - \mathbf{\Gamma})\mathbf{u}(t) - \mathbf{\Phi}\hat{\mathbf{x}}(t).\end{aligned}$$

- Letting $\mathbf{\Phi} = \mathbf{A} - \mathbf{L}\mathbf{C}$ and $\mathbf{\Gamma} = \mathbf{B} - \mathbf{L}\mathbf{D}$ results in

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t), \quad \mathbf{e}_0 = \mathbf{e}(0) = \mathbf{x}_0 - \hat{\mathbf{x}}_0.$$

- Thus, $\mathbf{e}(t) = \exp\left((\mathbf{A} - \mathbf{L}\mathbf{C})t\right)\mathbf{e}_0$, and $\bar{\mathbf{e}} = \mathbf{0}$ is an AS EP iff $\operatorname{Re}\left\{\lambda_i(\mathbf{A} - \mathbf{L}\mathbf{C})\right\} < 0$, $i = 1, 2, \ldots, n$, meaning that $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$.

- The observer dynamics are

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0), \tag{5}$$

  where $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)$.

- The term "$\mathbf{y}(t) - \hat{\mathbf{y}}(t)$" is called the residual or innovation. It's the error between the actual measurement and the predicted/estimated measurement.
- The key to having $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$, even when $\mathbf{A}$ is not Hurwitz, is **feedback**!
- The "$\mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t))$" term in the **closed-loop** observer dynamics of (5) is a feedback term.
  - When $\mathbf{y}(t) - \hat{\mathbf{y}}(t) = \mathbf{0}$ the true state $\mathbf{x}(t)$ and the estimated state $\hat{\mathbf{x}}(t)$ are identical; no need to "correct" the state estimate.
  - When $\mathbf{y}(t) - \hat{\mathbf{y}}(t) \neq \mathbf{0}$ the estimated state $\hat{\mathbf{x}}(t)$ needs to be "corrected"; the correction is proportional to $\mathbf{L}$.

- But wait . . .
- Is it always possible to place the eigenvalues of $\mathbf{A} - \mathbf{LC}$ in the OLHP?
- Let’s look at some examples.

Example

- Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{6}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0u(t). \tag{7}$$

- The observer is

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \right). \tag{8}$$

- The error dynamics are then

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} -\ell_1 & 1 \\ -\ell_2 & 1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}. \tag{10}$$

- The eigenvalues of $\mathbf{A} - \mathbf{LC}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda \mathbf{1} - (\mathbf{A} - \mathbf{LC})\right) && (11)\\
&= \det \begin{bmatrix} \lambda + \ell_1 & -1 \\ \ell_2 & \lambda - 1 \end{bmatrix} && (12)\\
&= (\lambda + \ell_1)(\lambda - 1) + \ell_2 && (13)\\
&= \lambda^2 + (\ell_1 - 1)\lambda + (-\ell_1 + \ell_2). && (14)
\end{aligned}
$$

- Recall that the roots of the second-order polynomial $a_2\lambda^2 + a_1\lambda + a_0 = 0$ are strictly in the OLHP iff $a_i > 0$, $i = 1, 2, 3$.
- Thus, pick $\ell_1 > 1$ and $-\ell_1 + \ell_2 > 0$ and the eigenvalues of $\mathbf{A} - \mathbf{LC}$ will be in the OLHP! Great!

- Say the desired eigenvalues of $\mathbf{A} - \mathbf{LC}$ are the roots of $\alpha(\lambda) = (\lambda + 2)(\lambda + 4) = \lambda^2 + 6\lambda + 8$.
- Pick $\ell_1 = 7$ and $\ell_2 = 15$.
- This is *eigenvalue placement*. In classical control (i.e., MECH 412) this is called *pole placement*.

Example

- Consider the system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \tag{15}$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + 0u(t). \tag{16}$$

- The observer is

$$\begin{bmatrix}\dot{\hat{x}}_1(t)\\ \dot{\hat{x}}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) + \begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} - \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}\right). \tag{17}$$

- The error dynamics are then

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \left(\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right)\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \tag{18}$$

$$= \begin{bmatrix}0 & 1-\ell_1\\ 0 & 1-\ell_2\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}. \tag{19}$$

- The eigenvalues of $\mathbf{A} - \mathbf{LC}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda \mathbf{1} - (\mathbf{A} - \mathbf{LC})\right) && (20)\\
&= \det \begin{bmatrix} \lambda & -(1-\ell_1) \\ 0 & \lambda - (1-\ell_2) \end{bmatrix} && (21)\\
&= \lambda(\lambda - (1-\ell_2)). && (22)
\end{aligned}
$$

- Although one eigenvalue can be placed at $\lambda_1 = 1 - \ell_2$, the other eigenvalue is $\lambda_2 = 0$ no matter what!
- Thus, the EP $\bar{\mathbf{e}} = \mathbf{0}$ of the closed-loop system cannot be rendered AS. Yikes!

Example

- Consider the system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \tag{23}$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + 0u(t). \tag{24}$$

- The observer is

$$\begin{bmatrix}\dot{\hat{x}}_1(t)\\ \dot{\hat{x}}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) + \begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} - \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}\right). \tag{25}$$

- The error dynamics are then

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \left(\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix} - \begin{bmatrix}\ell_1\\ \ell_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \tag{26}$$

$$= \begin{bmatrix}1-\ell_1 & 0\\ 1-\ell_2 & -1\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}. \tag{27}$$

- The eigenvalues of $\mathbf{A} - \mathbf{LC}$ are the roots of

$$
\begin{aligned}
0 &= \det\left(\lambda \mathbf{1} - (\mathbf{A} - \mathbf{LC})\right) && (28)\\
&= \det \begin{bmatrix} \lambda - (1 - \ell_1) & 0 \\ -1 + \ell_2 & \lambda + 1 \end{bmatrix} && (29)\\
&= (\lambda - (1 - \ell_1))(\lambda + 1). && (30)
\end{aligned}
$$

- Although one eigenvalue can be placed at $\lambda_1 = 1 - \ell_1$, the other eigenvalue is $\lambda_2 = -1$ no matter what.
- Thus, the EP $\bar{\mathbf{e}} = \mathbf{0}$ can be rendered AS, but the eigenvalues of $\mathbf{A} - \mathbf{LC}$ cannot be arbitrary placed in the OLHP.

Observability

- What conditions on the pair $(\mathbf{A}, \mathbf{C})$ must be met to allow for the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ to be
  - arbitrarily placed in the OLHP, or
  - partially placed in the OLHP,

  via proper design of $\mathbf{L}$?
- The case of arbitrary eigenvalue placement leads to the definition of *observability*.

Definition

$(\mathbf{A}, \mathbf{C})$ is *observable* if the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be arbitrarily assigned (with complex eigenvalues appearing in complex-conjugate pairs) by a suitable choice of $\mathbf{L}$.

- There are many ways to assess if $(\mathbf{A}, \mathbf{C})$ is observable.

Theorem

The following are equivalent.

- $(\mathbf{A}, \mathbf{C})$ *is observable.*
- *The observability matrix*

$$\mathbf{Q}_{\mathrm{o}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n_x - 1} \end{bmatrix} \tag{31}$$

  is full-column rank, $\operatorname{rank}\mathbf{Q}_{\mathrm{o}} = n_x$*.*
- *The observability Gramian*

$$\mathbf{W}_{\mathrm{o}}(t) = \int_0^t \exp(\mathbf{A}^{\mathsf{T}}\tau)\mathbf{C}^{\mathsf{T}}\mathbf{C}\exp(\mathbf{A}\tau)\,\mathrm{d}\tau \tag{32}$$

  is positive definite $\forall t > 0$*.*
- *The matrix* $\begin{bmatrix} \mathbf{A} - \lambda\mathbf{1} \\ \mathbf{C} \end{bmatrix}$ *has full-column rank* $\forall \lambda \in \mathbb{C}$*.*
- *Let* λ *and* $\mathbf{v}$ *be any right eigenpair of* $\mathbf{A}$ *(i.e.,* $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$*). Then* $\mathbf{C}\mathbf{v} \neq \mathbf{0}$*.*

In practice, use

```
Qo = control.obsv(A, C)
```

```
Qo = control.ctrb(A.T, C.T)
```

```
Wc = control.gram(P, 'o')
```

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{33}$$

- The observability matrix is

$$\mathbf{Q}_\mathrm{o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{34}$$

- $\mathrm{rank}\mathbf{Q}_\mathrm{o} = 2 = n_x$.
- Thus, the system is observable, meaning the eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be assigned arbitrarily.
- This result jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{35}$$

- The observability matrix is

$$\mathbf{Q}_\mathrm{o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{36}$$

- $\mathrm{rank}\mathbf{Q}_\mathrm{o} = 1 \neq n_x = 2.$
- Thus, the system is not observable, meaning the eigenvalues of $\mathbf{A} - \mathbf{LC}$ cannot be assigned arbitrarily.
- This result also jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{37}$$

- The observability matrix is

$$\mathbf{Q}_\mathrm{o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{38}$$

- $\mathrm{rank}\mathbf{Q}_\mathrm{o} = 1 \neq n_x = 2$.
- Thus, the system is not observable, meaning the eigenvalues of $\mathbf{A} - \mathbf{LC}$ cannot be assigned arbitrarily.
- This result, again, jives with our previous findings.

Detectability

- The case of partial eigenvalue placement where the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are in the OLHP, but not all eigenvalues are arbitrarily placed, leads to the definition of *detectability*.

Definition

$(\mathbf{A}, \mathbf{C})$ is *detectable* if there exists $\mathbf{L} \in \mathbb{R}^{n_x \times n_y}$ such that $\mathrm{Re}\left\{\lambda_i(\mathbf{A} - \mathbf{LC})\right\} < 0$, $i = 1, 2, \ldots, n_x$.

- There are many ways to assess if $(\mathbf{A}, \mathbf{C})$ is detectable.

Theorem

The following are equivalent.

- $(\mathbf{A}, \mathbf{C})$ *is detectable.*
- *The matrix* $\begin{bmatrix} \mathbf{A} - \lambda \mathbf{1} \\ \mathbf{C} \end{bmatrix}$ *has full-column rank* $\forall \mathrm{Re}\left\{\lambda\right\} \geq 0$.
- *For all* λ *and* $\mathbf{v}$ *such that* $\mathbf{Av} = \lambda \mathbf{v}$ *and* $\mathrm{Re}\left\{\lambda\right\} \geq 0$, $\mathbf{Cv} \neq \mathbf{0}$.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{39}$$

- The $\mathbf{A}$ matrix eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$.
- For $\lambda_1 = 0$, the matrix

$$\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{1} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 \\ 0 & 1 - \lambda_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{40}$$

  has column-rank equal to 1 (i.e., is not full-column rank).
- Thus, the system is *not* detectable.
  - No need to use λ_2 to assess detectability. Analysis with λ_1 indicated the system is not detectable.
- Again, this result jives with our previous findings.

Example

- Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{41}$$

- The $\mathbf{A}$ matrix eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.
- For $\lambda_1 = 1$, the matrix

$$\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{1} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} 1 - \lambda_1 & 0 \\ 1 & -1 - \lambda_1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 1 & 0 \end{bmatrix} \tag{42}$$

  has column-rank equal to 2 (i.e., is full-column rank).
- Thus, the system is detectable, there exists a $\mathbf{L}$ such that all the eigenvalues of $\mathbf{A} - \mathbf{LC}$ will be in the OLHP.
- Again, the jive is real . . .

Observer Design Example

- Consider the plant

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -1/10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{43}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0u(t). \tag{44}$$

  This plant can be interpreted as a mass-spring-damper system.
- Notice that $\mathbf{y}(t) \neq \mathbf{x}(t)$.
- When the plant is interpreted as a mass-spring-damper system, position is being measured.

Q. Design an observer so that with the slowest dominant eigenvalue of $\mathbf{A} - \mathbf{LC}$ at -2.

A. First, using `control.obsv`, check the pair $(\mathbf{A}, \mathbf{C})$ is observable, which is true.

- Next, design the observer gain $\mathbf{L}$ using `control.place`.
  - Place the eigenvalues of $\mathbf{A} - \mathbf{LC}$ at $-2, -3$ so that the slowest dominant closed-loop eigenvalue is -2.

- Simulate both the plant and the observer as an augmented system.
  - The plant,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{2}$$

feeds into the observer through the measurement $\mathbf{y}(t)$,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0), \tag{5}$$

where $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$.
  - Moreover, the input $\mathbf{u}(t)$ is applied to both the plant and the observer.
  - Thus, must simulate the interaction between the plant and the controller.
- The augmented system is

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{LC} & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} \mathbf{u}(t), \qquad \begin{bmatrix} \mathbf{x}(0) \\ \hat{\mathbf{x}}(0) \end{bmatrix}. \tag{45}$$

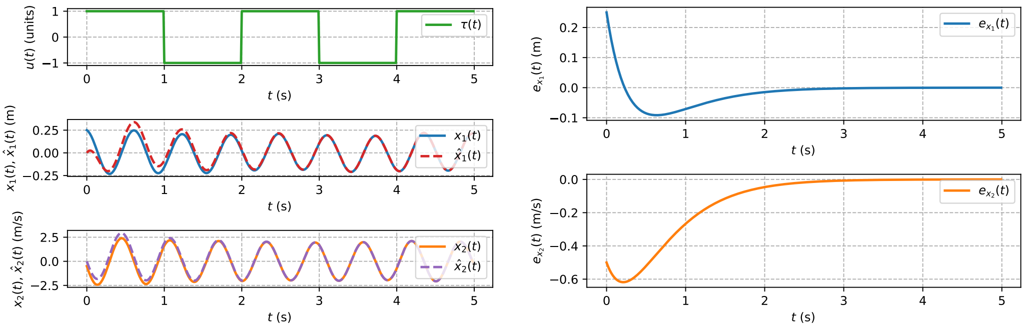

Figure 1: (Left) $u(t)$, $x_1(t)$, and $x_2(t)$ versus time. (Right) Estimation error versus time.

- Consider designing $\mathbf{L}$ by placing the eigenvalues of $\mathbf{A} - \mathbf{LC}$ at -20, -30.
- The state estimate $\hat{\mathbf{x}}(t)$ converges to $\mathbf{x}(t)$ much faster.

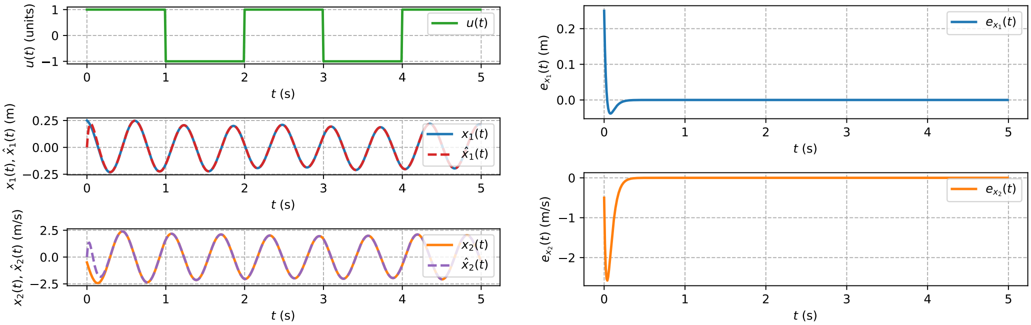

Figure 2: (Left) $u(t)$, $x_1(t)$, and $x_2(t)$ versus time. (Right) Estimation error versus time.

Velocity Measurement

- Consider measuring the velocity, so that $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t)$.
- The pair $(\mathbf{A}, \mathbf{C})$ is observable.
- Design $\mathbf{L}$ by placing the eigenvalues of $\mathbf{A} - \mathbf{LC}$ at -2, -3.

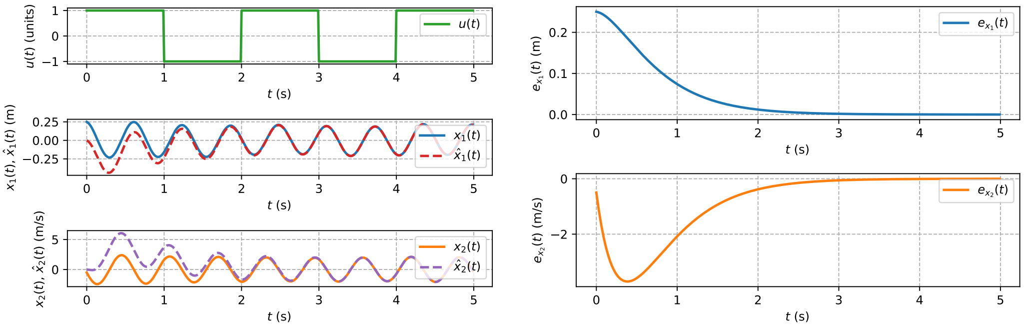

Figure 3: (Left) $u(t)$, $x_1(t)$, and $x_2(t)$ versus time. (Right) Estimation error versus time.

- Again, consider measuring the velocity, so that $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t)$.
- Design $\mathbf{L}$ by placing the eigenvalues of $\mathbf{A} - \mathbf{LC}$ at -20, -30.

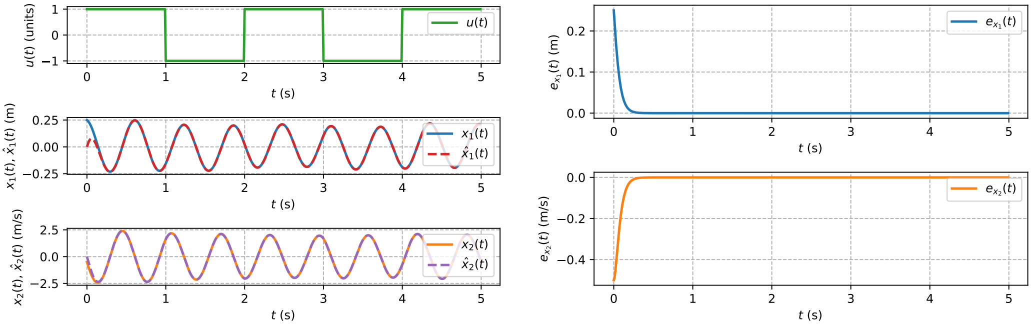

Figure 4: (Left) $u(t)$, $x_1(t)$, and $x_2(t)$ versus time. (Right) Estimation error versus time.

```python
import numpy as np
import control

# Plant
A = np.array([[0, 1],
              [-100, -0.1]])
B = np.array([[0],
              [1]])
C = np.array([[1, 0]])
D = np.array([[0],
              [0]])
x0 = np.array([0.25, -0.5])  # Initial conditions
n_x, n_u, n_y = A.shape[0], B.shape[1], C.shape[0]  # dimensions
P = control.ss(A, B, C, D)

# Observer gain design
desired_eigs_L = np.array([-2, -3])
L = control.place(A.T, C.T, desired_eigs_L)

```

Took out 2025/01/05

Theorem

Let $\mathbf{G}$ and $\mathbf{T}$ be any square, full rank, matrices of appropriate dimension. The following are equivalent.

- $(\mathbf{A}, \mathbf{C})$ *is detectable.*
- $(\mathbf{T}^{-1}(\mathbf{A} - \mathbf{LC})\mathbf{T}, \mathbf{GCT})$ *is detectable.*

References

These slides are based on [1–3].

[1] R. L. Williams and D. A. Lawrence, *Linear State-Space Control Systems*. Hoboken, NJ: John Wiley & Sons, Inc., 2007.

[2] J. Hespanha, *Linear Systems Theory*. Princeton, NJ: Princeton University Press, 2009.

[3] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, NJ: Prentice Hall, 1998.