Observer-Based Feedback Control

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- Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \tag{2}$$

  where $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{C})$ is observable (i.e., the state-space realization is **minimal**).
  - For simplicity of discussion, and to ensure the feedback interconnection of the plant and the controller is well posed, assume $\mathbf{D} = \mathbf{0}$.
  - Well posedness will discussed in detail later.
- Static full-state feedback,

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t),$$

  is generally not possible because $\mathbf{y}(t) \neq \mathbf{x}(t)$.
- Instead, consider an *observer-based controller*,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0), \tag{3}$$

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t). \tag{4}$$

- Design $\mathbf{K}$ and $\mathbf{L}$ to that $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$ are Hurwitz.

- But . . . will this work? Will the combination of the plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{2}$$

and the controller

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{y}(t), \quad \hat{\mathbf{x}}_0 = \hat{\mathbf{x}}(0), \tag{5}$$

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t). \tag{4}$$

in feedback be *closed-loop* asymptotically stable (CLAS)?

- Better find out . . .

- The plant dynamics are

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t).\end{aligned}$$

- The observer-based controller dynamics are

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{L}(\mathbf{C}\mathbf{x}(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \\ &= \mathbf{L}\mathbf{C}\mathbf{x}(t) + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t),\end{aligned}$$

  where $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$.

- The closed-loop dynamics are

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}) \end{bmatrix}}_{\mathbf{A}_{\mathrm{cl}}} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{cl}}}, \quad \mathbf{x}_{\mathrm{cl},0} = \begin{bmatrix} \mathbf{x}_0 \\ \hat{\mathbf{x}}_0 \end{bmatrix}$$

- Where are the $2n$ eigenvalues of $\mathbf{A}_{\mathrm{cl}}$ located? If the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$ are in the OLHP, does that mean the eigenvalues of $\mathbf{A}_{\mathrm{cl}}$ are in the OLHP?

- Consider a similarity (coordinate) transformation of the form

$$\mathbf{T}_{\mathrm{cl}}^{-1}\mathbf{A}_{\mathrm{cl}}\mathbf{T}_{\mathrm{cl}}$$

where

$$\mathbf{T}_{\mathrm{cl}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}.$$

- Note that $\mathbf{T}_{\mathrm{cl}} = \mathbf{T}_{\mathrm{cl}}^{-1}$ in this case, and that

$$\underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{cl}}} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}}_{\mathbf{z}}, \quad \text{or} \quad \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}}_{\mathbf{T}^{-1}} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}}_{\mathbf{x}_{\mathrm{cl}}}$$

- Also, recall that a similarity transformation does not alter eigenvalues. Thus, $\mathbf{A}_{\mathrm{cl}}$ and $\mathbf{T}_{\mathrm{cl}}^{-1}\mathbf{A}_{\mathrm{cl}}\mathbf{T}_{\mathrm{cl}}$ have the same spectrum (same eigenvalues)!

- $\mathbf{T}_{\mathrm{cl}}^{-1}\mathbf{A}_{\mathrm{cl}}\mathbf{T}_{\mathrm{cl}}$ is

$$\begin{aligned}\mathbf{T}_{\mathrm{cl}}^{-1}\mathbf{A}_{\mathrm{cl}}\mathbf{T}_{\mathrm{cl}} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & (\mathbf{A}-\mathbf{BK}-\mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{A}-\mathbf{BK} & -(\mathbf{A}-\mathbf{BK}-\mathbf{LC}) \end{bmatrix} = \begin{bmatrix} (\mathbf{A}-\mathbf{BK}) & \mathbf{BK} \\ \mathbf{0} & (\mathbf{A}-\mathbf{LC}) \end{bmatrix}\end{aligned}$$

- Note that

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{A}-\mathbf{BK}) & \mathbf{BK} \\ \mathbf{0} & (\mathbf{A}-\mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}.$$

- The eigenvalues of $\mathbf{T}_{\mathrm{cl}}^{-1}\mathbf{A}_{\mathrm{cl}}\mathbf{T}_{\mathrm{cl}}$, and of $\mathbf{A}_{\mathrm{cl}}$, are therefore the union of the eigenvalues of $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$.
- Thus, by designing $\mathbf{K}$ and $\mathbf{L}$ such that the eigenvalues of $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$ are in the OLHP means the eigenvalues associated with $\mathbf{A}_{\mathrm{cl}}$ are in the OLHP.
- In fact, the eigenvalues associated with $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$ can be designed *independent* of each other. This is referred to as the **separation property** associated with observer-based controllers.

Eigenvalue (Pole) Placement Design

- In practice, use `control.place` to design $\mathbf{K}$ and $\mathbf{L}$ such that the eigenvalues of $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$ are in the OLHP.
- One possible design procedure is as follows.
  - Design $\mathbf{K}$ so that $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ has desired eigenvalues.
    - Often tune eigenvalues by looking at the time-domain response (e.g., the transient).
  - Design $\mathbf{L}$ so that $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}(t)$, $\mathbf{e}(0) = \mathbf{e}_0$ satisfies

$$\max_{i=1,\dots n_x} \mathrm{Re}\left\{\lambda_i(\mathbf{A} - \mathbf{LC})\right\} \leq 4 \left(\min_{i=1,\dots,n_x} \mathrm{Re}\left\{\lambda_i(\mathbf{A} - \mathbf{BK})\right\} \right).$$

Example

- Consider the plant

$$P(s) = \frac{s-1}{s(s-2)(s^2-10s+26)}.$$

Q. Design an observer based compensator with the slowest dominant pole at -1.

A. First, find a minimal state-space realization, using `control.tf2ss`.

- Next, design the static full-state feedback gain $\mathbf{K}$ using `control.place`.
  - Place the eigenvalues of $\mathbf{A} - \mathbf{BK}$ at $-1, -1.5, -1.75, -2$.

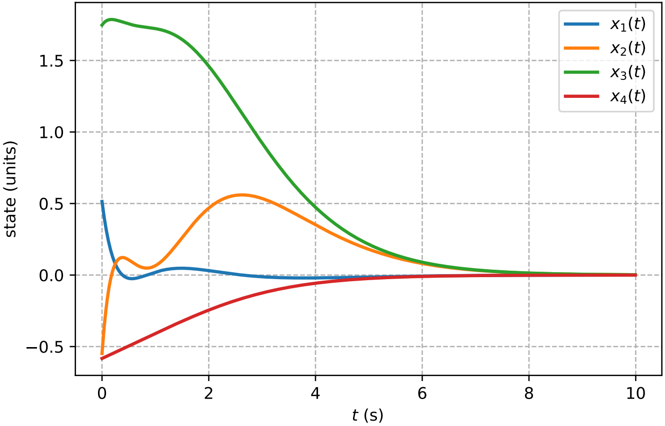

Figure 1: Response from random IC using static full-state feedback.

- Now, design the observer gain $\mathbf{L}$ using `control.place`.
  - Place the eigenvalues of $\mathbf{A} - \mathbf{LC}$ at $-8.0, -8.125, -8.25, -8.375$.

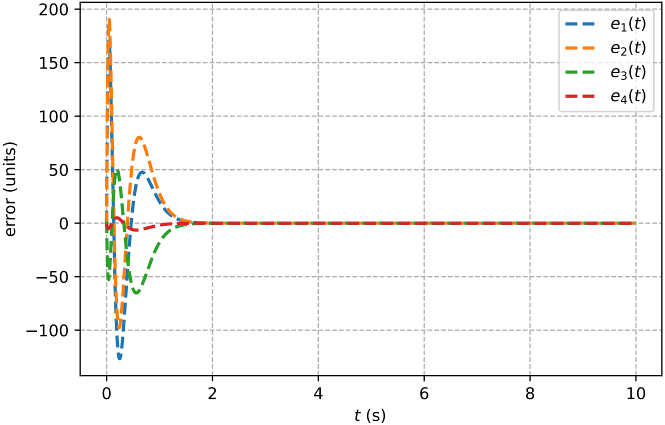

Figure 2: Error response of the observer from random IC.

- Form the closed-loop system (i.e., control the plant).

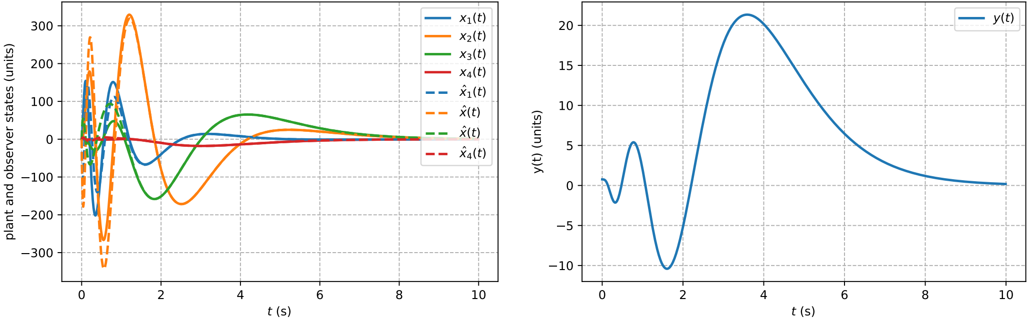

Figure 3: (Left) Plant and observer states vs. time. (Right) Output $y(t)$ versus time.

```python
import numpy as np
import control

s = control.tf('s')  # Laplace variable
P = (s - 1) / (s * (s - 2) * (s**2 - 10 * s + 26))  # Plant
P_ss = control.tf2ss(np.array(P.num).ravel(), np.array(P.den).ravel())  # ss form
A, B, C, D = P_ss.A, P_ss.B, P_ss.C, P_ss.D  # extract ss matrices
n_x, n_u, n_y = A.shape[0], B.shape[1], C.shape[0]  # dimensions

# Full-state feedback gain design
desired_eigs_K = np.array([-1, -1.5, -1.75, -2])
K = control.place(A, B, desired_eigs_K)

# Observer gain desgin
desired_eigs_L = np.array([-8.0, -8.125, -8.25, -8.375])
L = (control.place(A.T, C.T, desired_eigs_L)).T

# Controller
A_K, B_K, C_K, D_K = A - B @ K - L @ C, L, -K, np.zeros((n_y, n_y))
C_ss = control.ss(A_c, B_c, C_c, D_c)

```

- The closed-loop eigenvalues, $\lambda_i(\mathbf{A}_{\mathrm{cl}})$, $i = 1, \ldots, 2n_x$, are

$$-1, -1.5, -1.75, -2, -8.0, -8.125, -8.25, -8.375,$$

  as expected.
- The controller eigenvalues, $\lambda_i(\mathbf{A} - \mathbf{BK} - \mathbf{LC})$, $i = 1, \ldots, n_x$, are

$$-31.7, -10.6 \pm 22.4\jmath, 1.8$$

  - Notice the controller has a pole at $+1.8$; the controller is *unstable*!
  - Moreover, the controller has ORHP zeros.

General Dynamic Output Feedback Control

- Observer-based control, which is a particular kind of dynamic output feedback control, is of the form

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{y}(t),\\ \mathbf{u}(t) &= -\mathbf{K}\hat{\mathbf{x}}(t).\end{aligned}$$

- In general, a controller of the form

$$\dot{\mathbf{x}}_K(t) = \mathbf{A}_K\mathbf{x}_K(t) + \mathbf{B}_K\mathbf{u}_K(t), \tag{6}$$

$$\mathbf{y}_K(t) = \mathbf{C}_K\mathbf{x}_K(t), \tag{7}$$

  where $\mathbf{u}_K(t) = \mathbf{y}(t)$ and $\mathbf{u}(t) = -\mathbf{y}_K(t)$ is permissable.
- To be explicitly clear, a dynamic output feedback controller **does not have to be observer-based**.
- Any controller of the form given in (6) and (7) that renders the closed-loop dynamics matrix $\mathbf{A}_{\mathrm{cl}}$ Hurwitz, where

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}_K(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_K \\ -\mathbf{B}_K\mathbf{C} & \mathbf{A}_K \end{bmatrix}}_{\mathbf{A}_{\mathrm{cl}}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_K(t) \end{bmatrix}, \quad \mathbf{x}_{\mathrm{cl},0} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_{K,0} \end{bmatrix}$$

  is perfectly acceptable.
- Letting $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{x} \in \mathbb{R}^{n_K}$, it is not required that $n_K = n_x$, nor that $n_K \leq n_x$.
  - For example, a PID controller does not, in general, satisfy $n_K = n_x$.

References

See [1, Ch. 3], [2, Ch. 8], and J. How, “Dynamic Output-Feedback Control Architectures Matter”, IEEE CSM, 2016.

[1] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, NJ: Prentice Hall, 1998.

[2] R. L. Williams and D. A. Lawrence, *Linear State-Space Control Systems*. Hoboken, NJ: John Wiley & Sons, Inc., 2007.