The Invariant EKF Framework

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The Invariant EKF (InEKF) Framework

- State-estimation (navigation) problems found in robotics and aerospace engineering involve a state that is an element of a matrix Lie group.
- The InEKF framework is particularly well suited to state-estimation problems involving matrix Lie group states.
- The InEKF framework has four conditions that must be met for it to be (exactly) applicable to a state-estimation problem.

1. The measurement model must be either left or right invariant.
2. The invariant error must be consistent with the left or right invariance of the measurement model.
3. The invariant innovation must be consistent with the left or right invariance of the measurement model.
4. The process model must be group affine.

Definition (Invariant Measurement Model [1])

Consider a measurement $\mathbf{y}_k \in \mathbb{R}^n$, a state $\mathbf{X}_k \in G \subset \mathbb{R}^{n \times n}$, where G is a matrix Lie group, a known column matrix $\mathbf{b} \in \mathbb{R}^n$, and zero-mean Gaussian white noise $\mathbf{v}_k \in \mathbb{R}^n$. A left-invariant measurement model is of the form

$$\mathbf{y}_k^{\mathrm{L}} = \mathbf{X}_k \mathbf{b} + \mathbf{v}_k, \tag{1}$$

while a right-invariant measurement model is fo the form

$$\mathbf{y}_k^{\mathrm{R}} = \mathbf{X}_k^{-1} \mathbf{b} + \mathbf{v}_k. \tag{2}$$

- The measurement model (1) is left-invariant because, in the absence of noise,

$$\mathbf{y}_k^{\mathrm{L}} = \mathbf{X}_k \mathbf{b}_k, \tag{3}$$

$$\mathbf{X}'_k \mathbf{y}_k^{\mathrm{L}} = \mathbf{X}'_k \left(\mathbf{X}_k \mathbf{b}_k \right), \tag{4}$$

$$\tilde{\mathbf{y}}_k^{\mathrm{L}} = \tilde{\mathbf{X}}_k \mathbf{b}_k, \tag{5}$$

  where $\mathbf{X}'_k$ is arbitrary element of G.
- Typically measurements resolved in $\mathcal{F}_a$ are left-invariant.
- For example, the position measurement

$$\begin{bmatrix} \mathbf{y}_k \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_a^{z_k w} + \mathbf{v}_{a_k} \\ 1 \end{bmatrix} = \mathbf{T}_k \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{a_k} \\ 0 \end{bmatrix}, \tag{6}$$

  where

$$\mathbf{T}_k = \begin{bmatrix} \mathbf{C}_{ab_k} & \mathbf{r}_a^{z_k w} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3), \tag{7}$$

  is left invariant.
- This measurement could computed from GPS data, for example.

- The measurement model (2) is right-invariant because, in the absence of noise,

$$\mathbf{y}_k^{\mathrm{R}} = \mathbf{X}_k^{-1}\mathbf{b}_k, \tag{8}$$

$$\mathbf{X}_k'^{-1}\mathbf{y}_k^{\mathrm{R}} = \mathbf{X}_k'^{-1}\left(\mathbf{X}_k^{-1}\mathbf{b}_k\right) \tag{9}$$

$$= \left(\mathbf{X}_k\mathbf{X}_k'\right)^{-1}\mathbf{b}_k, \tag{10}$$

$$\tilde{\mathbf{y}}_k^{\mathrm{R}} = \tilde{\mathbf{X}}_k^{-1}\mathbf{b}_k. \tag{11}$$

  where $\mathbf{X}_k'$ is an arbitrary element of G.

- Typically measurements resolved in $\mathcal{F}_b$ are right-invariant.

- For example, let point p^i be associated with the i^{th} landmark. The position of the i^{th} landmark relative to point w resolved in $\mathcal{F}_a$ is $\mathbf{r}_a^{p_i w}$. The distance $\mathbf{r}_{b_k}^{p_i z}$ is measured (or computed). This measurement can be written

$$\begin{bmatrix} \mathbf{y}_k^i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{ab_k}^{\mathsf{T}}\left(\mathbf{r}_a^{p_i w} - \mathbf{r}_a^{z_k w}\right) + \mathbf{v}_{b_k}^i \\ 1 \end{bmatrix} = \mathbf{T}_k^{-1}\begin{bmatrix} \mathbf{r}_a^{p_i w} \\ 1 \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{b_k}^i \\ 0 \end{bmatrix}, \tag{12}$$

  where $\mathbf{v}_{b_k}^i \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k^i\right)$, $i = 1, \ldots, m$, where m is the number of landmarks.

- This measurement could be computed from stereo camera data, for example.

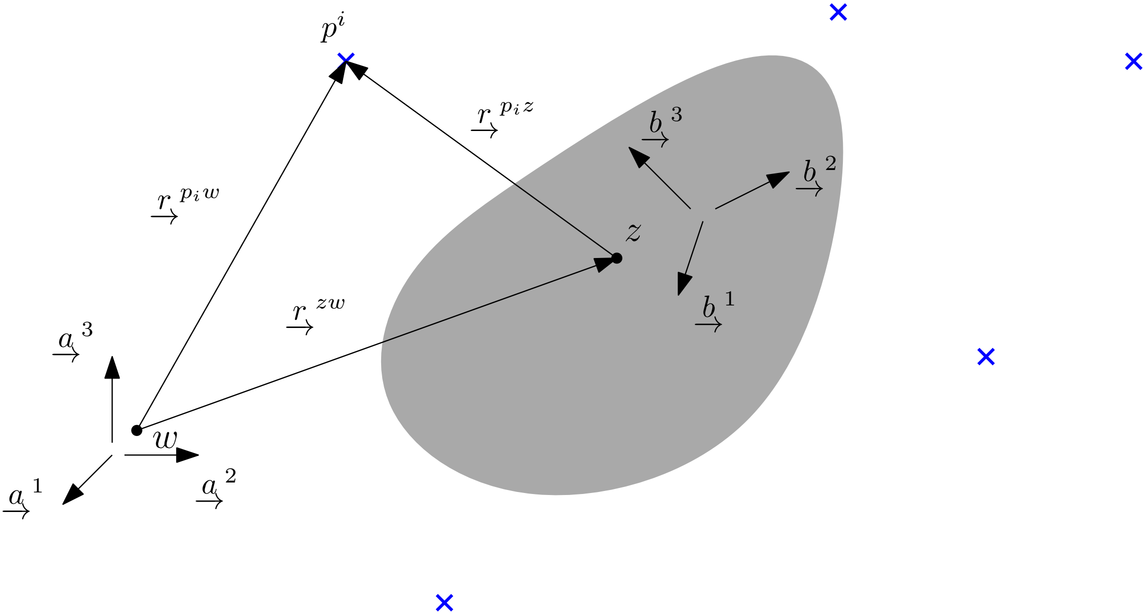

Figure 1: Measuring "$\mathbf{r}_{b_k}^{p_i z}$". Blue crosses are the observed landmarks.

Definition (Invariant Error [1])

Let $\mathbf{X}_k \in G$ be the true state of a system, $\bar{\mathbf{X}}_k \in G$ be a state different from the true state. The left-invariant error is defined as

$$\delta \mathbf{X}_k^{\mathrm{L}} = \mathbf{X}_k^{-1} \bar{\mathbf{X}}_k, \tag{13}$$

while the right-invariant error is defined as

$$\delta \mathbf{X}_k^{\mathrm{R}} = \bar{\mathbf{X}}_k \mathbf{X}_k^{-1}. \tag{14}$$

- In the InEKF, $\bar{\mathbf{X}}_k$ is either $\check{\mathbf{X}}_k$ or $\hat{\mathbf{X}}_k$.
- Can also work with $\mathbf{X}(t)$ and $\bar{\mathbf{X}}(t)$, leading to

$$\delta \mathbf{X}^{\mathrm{L}}(t) = \mathbf{X}(t)^{-1} \bar{\mathbf{X}}(t), \tag{15}$$
$$\delta \mathbf{X}^{\mathrm{R}}(t) = \bar{\mathbf{X}}(t) \mathbf{X}(t)^{-1}. \tag{16}$$

- Equation (13) is left-invariant,

$$\begin{aligned} \delta\mathbf{X}_k^{\mathrm{L}} &= \left(\mathbf{X}_k'\mathbf{X}_k\right)^{-1}\left(\mathbf{X}_k'\bar{\mathbf{X}}_k\right), \\ &= \mathbf{X}_k^{-1}\mathbf{X}_k'^{-1}\mathbf{X}_k'\bar{\mathbf{X}}_k \\ &= \mathbf{X}_k^{-1}\bar{\mathbf{X}}_k, \end{aligned} \tag{13}$$

  where $\mathbf{X}_k' \in G$ is arbitrary.

- Equation (14) is right-invariant,

$$\begin{aligned} \delta\mathbf{X}_k^{\mathrm{R}} &= \left(\bar{\mathbf{X}}_k\mathbf{X}_k'\right)\left(\mathbf{X}_k\mathbf{X}_k'\right)^{-1} \\ &= \bar{\mathbf{X}}_k\mathbf{X}_k'\mathbf{X}_k'^{-1}\mathbf{X}_k^{-1} \\ &= \bar{\mathbf{X}}_k\mathbf{X}_k^{-1}. \end{aligned} \tag{14}$$

  where $\mathbf{X}_k' \in G$ is arbitrary.

Definition (Invariant Innovation [1])

Let $\mathbf{y}_k, \check{\mathbf{y}}_k \in \mathbb{R}^{n_y}$ be the measurement and predicted measurement, respectively, which are either left- or right-invariant. When $\mathbf{y}_k, \check{\mathbf{y}}_k \in \mathbb{R}^{n_y}$ are left-invariant, the left-invariant innovation is

$$\mathbf{z}_k^{\mathrm{L}} = \check{\mathbf{X}}_k^{-1}(\mathbf{y}_k - \check{\mathbf{y}}_k), \tag{17}$$

while when $\mathbf{y}_k, \check{\mathbf{y}}_k \in \mathbb{R}^{n_y}$ are right-invariant, the right-invariant innovation is

$$\mathbf{z}_k^{\mathrm{R}} = \check{\mathbf{X}}_k(\mathbf{y}_k - \check{\mathbf{y}}_k). \tag{18}$$

- Using (13) and (17), the left-invariant innovation $\mathbf{z}_k^{\mathrm{L}}$ can be written as

$$\begin{aligned}\mathbf{z}_k^{\mathrm{L}} &= \check{\mathbf{X}}_k^{-1}(\mathbf{y}_k - \check{\mathbf{y}}_k) && (19)\\ &= \check{\mathbf{X}}_k^{-1}(\mathbf{X}_k\mathbf{b}_k + \mathbf{v}_k - \check{\mathbf{X}}_k\mathbf{b}_k) && (20)\\ &= \delta\check{\mathbf{X}}_k^{\mathrm{L}\,-1}\mathbf{b}_k + \check{\mathbf{X}}_k^{-1}\mathbf{v}_k - \mathbf{b}_k. && (21)\end{aligned}$$

- Notice how the left-invariant error appears in (21).
- To linearize (21), let

$$\mathbf{z}_k^{\mathrm{L}} = \bar{\mathbf{z}}_k^{\mathrm{L}} + \delta\mathbf{z}_k^{\mathrm{L}} = \delta\mathbf{z}_k^{\mathrm{L}}, \quad \delta\check{\mathbf{X}}_k^{\mathrm{L}\,-1} \approx \mathbf{1} - \delta\check{\boldsymbol{\xi}}_k^{\mathrm{L}\,\wedge}, \quad \mathbf{v}_k = \bar{\mathbf{v}}_k + \delta\mathbf{v}_k = \delta\mathbf{v}_k, \tag{22}$$

with $\bar{\mathbf{z}}_k^{\mathrm{L}} = \mathbf{0}$ and $\bar{\mathbf{v}}_k = \mathbf{0}$, resulting in

$$\begin{aligned}\mathbf{z}_k^{\mathrm{L}} &\approx \left(\mathbf{1} - \delta\check{\boldsymbol{\xi}}_k^{\mathrm{L}\,\wedge}\right)\mathbf{b}_k + \check{\mathbf{X}}_k^{-1}\delta\mathbf{v}_k - \mathbf{b}_k. && (23)\\ &= -\delta\check{\boldsymbol{\xi}}_k^{\mathrm{L}\,\wedge}\mathbf{b}_k + \check{\mathbf{X}}_k^{-1}\delta\mathbf{v}_k. && (24)\end{aligned}$$

- Lastly, (24) must be rearranged such that it can be written as

$$\mathbf{z}_k^{\mathrm{L}} = \mathbf{C}_k\delta\check{\boldsymbol{\xi}}_k^{\mathrm{L}} + \mathbf{M}_k\delta\mathbf{v}_k, \tag{25}$$

where $\mathbf{M}_k = \check{\mathbf{X}}_k^{-1}$. The measurement model Jacobian $\mathbf{C}_k$ *will only depend on the known vector* $\mathbf{b}_k$.

- Using (14) and (18), the right-invariant innovation $\mathbf{z}_k^{\mathrm{R}}$ can be written as

$$\mathbf{z}_k^{\mathrm{R}} = \check{\mathbf{X}}_k \left(\mathbf{y}_k - \check{\mathbf{y}}_k\right) \tag{26}$$
$$= \check{\mathbf{X}}_k \left(\mathbf{X}_k^{-1}\mathbf{b}_k + \mathbf{v}_k - \check{\mathbf{X}}_k^{-1}\mathbf{b}_k\right) \tag{27}$$
$$= \delta\check{\mathbf{X}}_k^{\mathrm{R}}\mathbf{b}_k - \mathbf{b}_k + \check{\mathbf{X}}_k\mathbf{v}_k. \tag{28}$$

- Notice how the right-invariant error appears in (28).
- To linearize (28), let

$$\mathbf{z}_k^{\mathrm{R}} = \bar{\mathbf{z}}_k^{\mathrm{R}} + \delta\mathbf{z}_k^{\mathrm{R}} = \delta\mathbf{z}_k^{\mathrm{R}}, \quad \delta\check{\mathbf{X}}_k^{\mathrm{R}} \approx \mathbf{1} + \delta\check{\boldsymbol{\xi}}_k^{\mathrm{R}\wedge}, \quad \mathbf{v}_k = \bar{\mathbf{v}}_k + \delta\mathbf{v}_k, \tag{29}$$

with $\bar{\mathbf{z}}_k^{\mathrm{R}} = \mathbf{0}$ and $\bar{\mathbf{v}}_k = \mathbf{0}$, resulting in

$$\mathbf{z}_k^{\mathrm{R}} \approx \left(\mathbf{1} + \delta\check{\boldsymbol{\xi}}_k^{\mathrm{R}\wedge}\right)\mathbf{b}_k - \mathbf{b}_k + \check{\mathbf{X}}_k\delta\mathbf{v}_k$$
$$= \delta\check{\boldsymbol{\xi}}_k^{\mathrm{R}\wedge}\mathbf{b}_k + \check{\mathbf{X}}_k\delta\mathbf{v}_k, \tag{30}$$

- Lastly, (30) must be rearranged such that it can be written as

$$\mathbf{z}_k^{\mathrm{R}} = \mathbf{C}_k\delta\check{\boldsymbol{\xi}}_k^{\mathrm{R}} + \mathbf{M}_k\delta\mathbf{v}_k, \tag{31}$$

where $\mathbf{M}_k = \check{\mathbf{X}}_k$. Once again, the measurement model Jacobian $\mathbf{C}_k$ is *state (estimate) independent*, as it only depends on the known vector $\mathbf{b}_k$.

Definition (Group Affine [1])

A function $\mathbf{F}(\mathbf{X}, \mathbf{u}) : G \times \mathbb{R}^{n_u} \to G$ is said to be group affine if for all $\mathbf{X}_1, \mathbf{X}_2 \in G$ and $\mathbf{u} \in \mathbb{R}^{n_u}$, the equation

$$\mathbf{F}(\mathbf{X}_1\mathbf{X}_2, \mathbf{u}) = \mathbf{X}_1\mathbf{F}(\mathbf{X}_2, \mathbf{u}) + \mathbf{F}(\mathbf{X}_1, \mathbf{u})\mathbf{X}_2 - \mathbf{X}_1\mathbf{F}(\mathbf{1}, \mathbf{u})\mathbf{X}_2 \tag{32}$$

is satisfied.

- Matrix Lie group kinematics of the form

$$\dot{\mathbf{X}} = \mathbf{X}\mathbf{u}^{\wedge} = \mathbf{F}(\mathbf{X}, \mathbf{u}), \tag{33}$$

  are generally group affine.

- Recall that an affine function $\mathbf{f} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ of the form
$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}^{n_x}, \ \mathbf{b} \in \mathbb{R}^{n_y}$$
is composed of two parts: the linear part, "$\mathbf{A}\mathbf{x}$", and the affine part, "$+\mathbf{b}$".
- To understand what group affine means [2], recall the following.
  - In $\mathbb{R}^{n_x}$ "linear" means (in part) "superposition" or "addition", that is
$$\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^{n_x}, \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{n_x},$$
while a group sense "linear" means "multiplication", that is
$$\mathbf{X}_1\mathbf{X}_2 \in G, \quad \forall \mathbf{X}_1, \mathbf{X}_2 \in G$$
  - An *automorphism* of the matrix Lie group G is an invertible function $\boldsymbol{A} : G \to G$ such that
$$\boldsymbol{A}(\mathbf{X}_1\mathbf{X}_2) = \boldsymbol{A}(\mathbf{X}_1)\boldsymbol{A}(\mathbf{X}_2), \quad \forall \mathbf{X}_1, \mathbf{X}_2 \in G.$$
  - Note that the definition of an automorphsim implies that
$$\boldsymbol{A}(\mathbf{X}^{-1}) = (\boldsymbol{A}(\mathbf{X}))^{-1}, \quad \boldsymbol{A}(\mathbf{1}) = \mathbf{1}.$$
- A group affine function $\mathbf{F} : G \to G$ of the form
$$\mathbf{F}(\mathbf{X}) = \boldsymbol{A}(\mathbf{X})\mathbf{b}^\wedge, \quad \mathbf{X} \in G, \quad \mathbf{b}^\wedge \in \mathfrak{g},$$
where $\boldsymbol{A} : G \to G$ is an automorphism, is composed of two parts: the group part, "$\boldsymbol{A}(\mathbf{X})$", and the affine part, "$\mathbf{b}^\wedge$".

- Consider
$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \mathbf{M}\mathbf{x} + \mathbf{u}, \tag{34}$$
an affine function, where $\mathbf{f} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$.
- Equation (34) is group affine where the group operation "$\cdot$" is now "$+$", and "identity" is "zero", that is,
$$\begin{aligned}
&\mathbf{X}_1 \cdot \mathbf{F}(\mathbf{X}_2,\mathbf{u}) + \mathbf{F}(\mathbf{X}_1,\mathbf{u}) \cdot \mathbf{X}_2 - \mathbf{X}_1 \cdot \mathbf{F}(\mathbf{1},\mathbf{u}) \cdot \mathbf{X}_2 && (35)\\
&= \mathbf{x}_1 + (\mathbf{M}\mathbf{x}_2 + \mathbf{u}) + (\mathbf{M}\mathbf{x}_1 + \mathbf{u}) + \mathbf{x}_2 - (\mathbf{x}_1 + (\mathbf{0} + \mathbf{u}) + \mathbf{x}_2) && (36)\\
&= \mathbf{M}(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{u} = \mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{u}) = \mathbf{F}(\mathbf{X}_1 \cdot \mathbf{X}_2, \mathbf{u}) && (37)
\end{aligned}$$
- Consider
$$\dot{\mathbf{X}} = \mathbf{X}\mathbf{u}^{\wedge} = \mathbf{F}(\mathbf{X},\mathbf{u}). \tag{33}$$
- Equation (33) is group affine, that is
$$\begin{aligned}
&\mathbf{X}_1\mathbf{F}(\mathbf{X}_2,\mathbf{u}) + \mathbf{F}(\mathbf{X}_1,\mathbf{u})\mathbf{X}_2 - \mathbf{X}_1\mathbf{F}(\mathbf{1},\mathbf{u})\mathbf{X}_2 && (38)\\
&= \mathbf{X}_1\mathbf{X}_2\mathbf{u}^{\wedge} + \mathbf{X}_1\mathbf{u}^{\wedge}\mathbf{X}_2 - \mathbf{X}_1\mathbf{1}\mathbf{u}^{\wedge}\mathbf{X}_2 && (39)\\
&= \mathbf{X}_1\mathbf{X}_2\mathbf{u}^{\wedge} && (40)\\
&= \mathbf{F}(\mathbf{X}_1\mathbf{X}_2,\mathbf{u}). && (41)
\end{aligned}$$

Process and Measurement Model

- Consider a process model of the form

$$\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}(t), \mathbf{u}(t)) + \mathbf{X}(t)\mathbf{W}(t), \tag{42}$$

  where $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is an input variable, $\mathbf{X}(t) \in G$ is the state, and $\mathbf{W}(t) \in \mathfrak{g}$ is process noise such that $\mathbf{W}(t)^\vee = \mathbf{w}(t) \in \mathcal{N}(\mathbf{0}, \boldsymbol{Q}\delta(t - \tau))$.
- The function $\mathbf{F}(\mathbf{X}(t), \mathbf{u}(t))$ is group affine.
  - Going forward, the argument of time, "(t)", will be suppressed.

- Consider a measurement model of the form

$$\mathbf{y}_k = \mathbf{g}(\mathbf{X}_k, \mathbf{v}_k), \tag{43}$$

  where $\mathbf{X}_k = \mathbf{X}(t_k) \in G$ is the state, and $\mathbf{v}_k \in \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$ is measurement noise.
- The function $\mathbf{g}(\mathbf{X}_k, \mathbf{v}_k)$ is either left- or right-invariant.

The left- or right-invariant form of the measurement model dictates which form the InEKF is used!

The InEKF Prediction Step

- Using either a left- or right-invariant error, linearize the process model, yielding
$$\delta\dot{\boldsymbol{\xi}}(t) = \mathbf{A}(t)\delta\boldsymbol{\xi}(t) + \mathbf{L}(t)\delta\mathbf{w}(t). \tag{44}$$
  - $\delta\mathbf{X}(t) = \exp(\delta\boldsymbol{\xi}(t)^\wedge)$, where $\delta\mathbf{X}$ is either a left- or right-invariant error.
  - $\mathbf{A}(t)$ depends on $\mathbf{u}(t)$ but is guaranteed to not depend on $\check{\mathbf{X}}(t)$,
  - There's no guarantee that $\mathbf{L}(t)$ will not depend on $\check{\mathbf{X}}(t)$. Sometimes it does, sometimes it doesn't.
- The continuous-time prediction is
$$\dot{\check{\mathbf{X}}}(t) = \mathbf{F}(\check{\mathbf{X}}(t), \mathbf{u}(t)), \tag{45}$$
$$\dot{\check{\mathbf{P}}}(t) = \mathbf{A}(t)\check{\mathbf{P}}(t) + \check{\mathbf{P}}(t)\mathbf{A}(t)^\mathsf{T} + \mathbf{L}(t)\boldsymbol{Q}\mathbf{L}(t)^\mathsf{T}, \tag{46}$$
which starts at $\check{\mathbf{X}}(t_{k-1}) = \hat{\mathbf{X}}_{k-1}$ and $\check{\mathbf{P}}(t_{k-1}) = \hat{\mathbf{P}}_{k-1}$, and ends at $\check{\mathbf{X}}_k = \check{\mathbf{X}}(t_k)$ and $\check{\mathbf{P}}_k = \check{\mathbf{P}}(t_k)$.
- Alternatively, the discrete-time prediction is
$$\check{\mathbf{X}}_k = \mathbf{F}_{k-1}(\hat{\mathbf{X}}_{k-1}, \mathbf{u}_{k-1}), \tag{47}$$
$$\check{\mathbf{P}}_k = \mathbf{A}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{A}_{k-1}^\mathsf{T} + \mathbf{Q}_{k-1}. \tag{48}$$
where $\mathbf{A}_{k-1}$ and $\mathbf{Q}_{k-1}$ are found using van Loan's method.

The InEKF Correction Step - Left-Invariant Case

- When using a left-invariant error, the error between the predicted state and the corrected state is $\delta \mathbf{X}_k^{\mathrm{L}} = \hat{\mathbf{X}}_k^{-1} \check{\mathbf{X}}_k$. Rearranging leads to

$$\delta \mathbf{X}_k^{\mathrm{L}} = \hat{\mathbf{X}}_k^{-1} \check{\mathbf{X}}_k, \tag{49}$$

$$\hat{\mathbf{X}}_k \delta \mathbf{X}_k^{\mathrm{L}} = \check{\mathbf{X}}_k, \tag{50}$$

$$\hat{\mathbf{X}}_k = \check{\mathbf{X}}_k \delta \mathbf{X}_k^{\mathrm{L}^{-1}} = \check{\mathbf{X}}_k \exp\left(\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{L}}\right)^{\wedge}\right)^{-1} = \check{\mathbf{X}}_k \exp\left(-\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{L}}\right)^{\wedge}\right). \tag{51}$$

- Thus, the left-invariant correction is

$$\hat{\mathbf{X}}_k = \check{\mathbf{X}}_k \exp\left(-\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{L}}\right)^{\wedge}\right), \tag{52}$$

  where

$$\mathbf{K}_k = \check{\mathbf{P}}_k \mathbf{C}_k^{\mathsf{T}} (\mathbf{C}_k \check{\mathbf{P}}_k \mathbf{C}_k^{\mathsf{T}} + \mathbf{M}_k \mathbf{R}_k \mathbf{M}_k^{\mathsf{T}})^{-1}. \tag{53}$$

- The Jacobians $\mathbf{C}_k$ and $\mathbf{M}_k$ in (53) come from linearization of the left-invariant innovation, as was shown in (24), leading to

$$\mathbf{z}_k^{\mathrm{L}} = \mathbf{C}_k \delta \check{\boldsymbol{\xi}}_k^{\mathrm{L}} + \mathbf{M}_k \delta \mathbf{v}_k. \tag{25}$$

The InEKF Correction Step - Right-Invariant Case

- When using a left-invariant error, the error between the predicted state and the corrected state is $\delta \mathbf{X}_k^{\mathrm{R}} = \check{\mathbf{X}}_k \hat{\mathbf{X}}_k^{-1}$. Rearranging leads to

$$\delta \mathbf{X}_k^{\mathrm{R}} = \check{\mathbf{X}}_k \hat{\mathbf{X}}_k^{-1}, \tag{54}$$

$$\delta \mathbf{X}_k^{\mathrm{R}} \hat{\mathbf{X}}_k = \check{\mathbf{X}}_k, \tag{55}$$

$$\hat{\mathbf{X}}_k = \delta \mathbf{X}_k^{\mathrm{R}^{-1}} \check{\mathbf{X}}_k \tag{56}$$

$$= \exp\left(\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{R}}\right)^{\wedge}\right)^{-1} \check{\mathbf{X}}_k \tag{57}$$

$$= \exp\left(-\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{R}}\right)^{\wedge}\right) \check{\mathbf{X}}_k. \tag{58}$$

- Thus, the right-invariant correction is

$$\hat{\mathbf{X}}_k = \exp\left(-\left(\mathbf{K}_k \mathbf{z}_k^{\mathrm{R}}\right)^{\wedge}\right) \check{\mathbf{X}}_k, \tag{59}$$

  where $\mathbf{K}_k$ is given in (53).

- The Jacobians $\mathbf{C}_k$ and $\mathbf{M}_k$ in $\mathbf{K}_k$ come from linearization of the right-invariant innovation, as was shown in (30), leading to

$$\mathbf{z}_k^{\mathrm{R}} = \mathbf{C}_k \delta \check{\boldsymbol{\xi}}_k^{\mathrm{R}} + \mathbf{M}_k \delta \mathbf{v}_k. \tag{31}$$

- In both the left- and right-invariant cases, the covariance update is

$$\hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{C}_k)\check{\mathbf{P}}_k(\mathbf{1} - \mathbf{K}_k\mathbf{C}_k)^{\mathsf{T}} + \mathbf{K}_k\mathbf{M}_k\mathbf{R}_k\mathbf{M}_k^{\mathsf{T}}\mathbf{K}_k^{\mathsf{T}}. \tag{60}$$

Summary

	Error Definition	Innovation	State Correction
KF/EKF	$\delta \mathbf{x} = \mathbf{x} - \hat{\mathbf{x}}$	$\mathbf{z}_k = \mathbf{y} - \check{\mathbf{y}}_k$	$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k \mathbf{z}_k$
LInEKF	$\delta \mathbf{X}^{\mathrm{L}} = \mathbf{X}^{-1} \hat{\mathbf{X}}$	$\mathbf{z}_k = \check{\mathbf{X}}_k^{-1} (\mathbf{y}_k - \check{\mathbf{y}}_k)$	$\hat{\mathbf{X}}_k = \check{\mathbf{X}}_k \exp\left(-(\mathbf{K}_k \mathbf{z}_k^{\mathrm{L}})^{\wedge}\right)$
RInEKF	$\delta \mathbf{X}^{\mathrm{R}} = \hat{\mathbf{X}} \mathbf{X}^{-1}$	$\mathbf{z}_k = \check{\mathbf{X}}_k (\mathbf{y}_k - \check{\mathbf{y}}_k)$	$\hat{\mathbf{X}}_k = \exp\left(-(\mathbf{K}_k \mathbf{z}_k^{\mathrm{R}})^{\wedge}\right) \check{\mathbf{X}}_k$

Table 1: Summary of the key KF/EKF, MEKF, LInEKF, RInEKF equations.

- The invariant framework guarantees that $\mathbf{A}$, or $\mathbf{A}_{k-1}$, and $\mathbf{C}_k$ will be state-estimate independent.
- However, invariant framework does not guarantee $\mathbf{L}$, or $\mathbf{L}_{k-1}$, and $\mathbf{M}_k$ will be state-estimate independent.

References

These slides are based on [1–3].

Insights into “what group affine means” are due to conversations with Paul Chauchat [2].

[1] A. Barrau and S. Bonnabel, “The Invariant Extended Kalman Filter as a Stable Observer,” *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, 2017.

[2] P. Chauchat, *Smoothing algorithms for navigation, localisation and mapping based on high-grade inertial sensors*. PhD Thesis, Mines ParisTech, 2020.

[3] A. Barrau, *Non-linear state error based extended Kalman filters with applications to navigation*. PhD Thesis, MINES ParisTech, Centre de Robotique, 2015.